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Renormalization and wandering continua of rational maps¹

Guizhen Cui, Wenjuan Peng and Lei Tan

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Abstract

Renormalizations can be considered as building blocks of complex dynamical systems. This phenomenon has been widely studied for iterations of polynomials of one complex variable. Concerning non-polynomial hyperbolic rational maps, a recent work of Cui-Tan shows that these maps can be decomposed into postcritically finite renormalization pieces. The main purpose of the present work is to perform the surgery one step deeper. Based on Thurston's idea of decompositions along multicurves, we introduce a key notion of Cantor multicurves (a stable multicurve generating infinitely many homotopic curves under pullback), and prove that any postcritically finite piece having a Cantor multicurve can be further decomposed into smaller postcritically finite renormalization pieces.

As a byproduct, we establish the presence of separating wandering continua in the corresponding Julia sets.

Contrary to the polynomial case, we exploit tools beyond the category of analytic and quasiconformal maps, such as Rees-Shishikura's semi-conjugacy for topological branched coverings that are Thurston-equivalent to rational maps.

1 Introduction

The combinatorics of polynomial dynamics have been intensively studied for many years. They may be described by kneading sequences in the real case ([20]), by Hubbard trees ([8]), Thurston's laminations ([27, 8, 18]), critical point portraits ([11, 4, 13]) or fixed point portraits ([13]) in the postcritically finite case, and by Yoccoz puzzles (see e.g. [28, 17, 18]) in the general case. Using these tools, one can study renormalizations of polynomials. They are first return maps on proper subsets of the Julia set behaving like other polynomials on their own Julia sets. For example a Douady rabbit may appear as a proper subset of the Julia set of a different polynomial. One can also show that the Julia set J of a polynomial does not possess wandering continua, provided that J is connected and locally connected (see Blokh & Levin, Kiwi, and Thurston [3, 14, 27]). Here a wandering continuum is a compact connected set with more than one point whose successive forward images are pairwise disjoint.

The combinatorics of non-polynomial rational map dynamics are much harder to study, due to the lack of an invariant Fatou basin to encode the combinatorial information. One of the fundamental results in this direction is Thurston's study on stable multicurves together with his topological characterization of postcritically finite rational maps (see for example Douady-Hubbard [9]).

As an example, it may happen that a rational map f pulls back some particular Jordan curve γ to a homotopic Jordan curve. Then γ on its own forms a stable multicurve, and the Julia set of f can be 'cut-open' along γ into two parts each

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being the Julia set of some appropriate polynomial. Such an f is called a **mating** of two polynomials. See for example [24, 25]. Notice that the two small Julia sets can not be considered as renormalizations of f in the classical sense, as most of the time they are not embedding in the Julia set of f and the related dynamical systems are not topologically conjugate.

A map is **sub-hyperbolic** if every critical point of it is either preperiodic or attracted by an attracting periodic cycle. Thurston's theory can be applied to study sub-hyperbolic rational maps with disconnected Julia sets. In this case one may expect to find a stable multicurve within the multiply-connected Fatou components to perform a decomposition. This is precisely what has been done by Cui-Tan in the work [6]. Furthermore, they proved that the decomposition gives rise to finitely many renormalization pieces, each being a postcritically finite rational map (in particular having a connected and locally connected Julia set).

A prototype example is a cubic polynomial with one escaping critical point and another critical point realizing a renormalization of a quadratic polynomial. A stable multicurve can be found through successive pullbacks of an equipotential in the basin of infinity.

The next step is to study the combinatorics of postcritically finite rational maps. For example one may ask if such a map is renormalizable (in the classical sense), or if it has a wandering continuum in its Julia set. As far as we know, few results in this direction are currently known.

The purpose of the present work is to solve some of these problems.

As we have seen in the mating case, if we decompose a connected Julia set along a random stable multicurve, we may obtain small Julia sets that touch each other in the big Julia set and do not provide suitable candidates for renormalizations.

Our key idea to overcome this difficulty is to introduce the notion of **Cantor multicurves**, meaning roughly multicurves whose successive pullbacks generate infinitely many homotopic curves. We will then be able to prove that if a postcritically finite rational map f has a Cantor multicurve Γ , then the small Julia sets of a decomposition along Γ are pairwise disjoint, leading therefore to renormalization pieces. Here a **renormalization** of f is a pair (f^p, J') for some $p \geq 1$ and some proper subset J' of the Julia set so that $f^p : J' \rightarrow J'$ is conjugate to the action of some postcritically finite rational map on its own Julia set.

In fact, we will prove that any Cantor multicurve Γ induces a collection of annuli \mathcal{A} within the homotopy class of Γ as well as a collection \mathcal{A}^1 of essential sub-annuli of \mathcal{A} , so that $\partial\mathcal{A} \subset \partial\mathcal{A}^1$ and $f : \mathcal{A}^1 \rightarrow \mathcal{A}$ is a covering. Moreover the underlying unweighted transition matrix is strongly expanding. Such an annular self-covering dynamical system will be called **an exact annular system**.

As a byproduct, we immediately derive the existence of wandering Jordan curves in the Julia set: they lie in the exact annular system.

Here are more precise statements that we will prove:

Theorem 1.1. *A postcritically finite rational map with a Cantor multicurve has an exact annular system homotopic to this Cantor multicurve rel the post-critical set.*

Theorem 1.2. *A postcritically finite rational map with a stable Cantor multicurve has a renormalization.*

Theorem 1.3. *A postcritically finite rational map has a Cantor multicurve if and only if it has a separating wandering continuum. Moreover, the wandering continuum is eventually a Jordan curve.*

Examples of rational maps with Cantor multicurves will be constructed in [7].

Theorem 1.1 is the main result of this work. To prove it, we first modify the rational map to a Thurston-equivalent branched covering having a topological exact annular system. Then applying a theorem due to Rees and Shishikura, we obtain a semi-conjugacy from the branched covering to the rational map. The key point in the proof is to show that the exact annular system is preserved under the semi-conjugacy.

We want to emphasize that the spirit of Thurston's theory such as stable multicurves, moduli of annuli, transition matrices etc are everywhere present in this work. However we shall not need to make use of the fundamentally deeper characterization theorem of Thurston, in particular no iterations in Teichmüller spaces. In this sense our result here remains elementary. It can also be considered as an introduction to Thurston's theory.

This manuscript is organized as follows. In §2, we give the definition of Cantor multicurves. We then give some equivalent conditions in the irreducible case. In §3, we introduce the notion of an exact annular system and show that every component of its Julia set is a Jordan curve if it is expanding. In §4, we prove Theorem 1.1. In §5 we define the renormalization of rational maps, then prove Theorem 1.2. In §6, we give the definition of separating wandering continua then prove Theorem 1.3. An example of rational maps with wandering continua is given in this section. Appendix A is devoted to the theorem of Rees-Shishikura.

2 Multicurves

Let $F : \widehat{\mathbb{C}} \rightarrow \widehat{\mathbb{C}}$ be a branched covering of the Riemann sphere $\widehat{\mathbb{C}}$. We always assume $\deg F \geq 2$ in this paper. Denote by Ω_F the critical point set of F . The **post-critical set** of F is defined by

$$\mathcal{P}_F = \bigcup_{n \geq 1} \overline{F^n(\Omega_F)}.$$

The map F is said to be **postcritically finite** if \mathcal{P}_F is finite. Refer to [9, 18, 27] for the following definitions.

Definition 1. Let $F : \widehat{\mathbb{C}} \rightarrow \widehat{\mathbb{C}}$ be a postcritically finite branched covering. We say that a Jordan curve γ on $\widehat{\mathbb{C}} \setminus \mathcal{P}_F$ is **non-essential** (resp. **peripheral**) if one component of $\widehat{\mathbb{C}} \setminus \gamma$ contains zero (resp. one) point of \mathcal{P}_F , or **non-peripheral** if each component of $\widehat{\mathbb{C}} \setminus \gamma$ contains at least two points of \mathcal{P}_F .

A **multicurve** of F is a finite non-empty collection of disjoint non-peripheral Jordan curves on $\widehat{\mathbb{C}} \setminus \mathcal{P}_F$ such that any two of them are not homotopic rel \mathcal{P}_F . A multicurve Γ is **stable** if each non-peripheral curve in $F^{-1}(\gamma)$ for $\gamma \in \Gamma$ is homotopic rel \mathcal{P}_F to a curve in Γ . A multicurve Γ is **pre-stable** if each curve $\gamma \in \Gamma$ is homotopic rel \mathcal{P}_F to a curve in $F^{-1}(\beta)$ for some curve $\beta \in \Gamma$.

A pre-stable multicurve Γ is **irreducible** if for each pair $(\gamma, \beta) \in \Gamma \times \Gamma$, there is an integer $n \geq 1$ such that $F^{-n}(\beta)$ has a component δ homotopic to γ rel \mathcal{P}_F and $F^k(\delta)$ is homotopic to a curve in Γ for $1 \leq k < n$.

Convention. For Γ a collection of curves in $\widehat{\mathbb{C}}$, we also use Γ to denote the union of curves in Γ as a subset of $\widehat{\mathbb{C}}$ if there is no confusion.

Let Γ be a multicurve of F . For each $\gamma \in \Gamma$, define $\Gamma(1, \gamma)$ to be the collection of curves in $F^{-1}(\gamma)$ homotopic rel \mathcal{P}_F to γ . Define $\Gamma(1, \Gamma) := \bigcup_{\gamma \in \Gamma} \Gamma(1, \gamma)$. Inductively, for $n \geq 1$, define $\Gamma(n+1, \gamma)$ to be the collection of curves in $F^{-1}(\Gamma(n, \Gamma))$

homotopic rel \mathcal{P}_F to γ and $\Gamma(n+1, \Gamma) := \bigcup_{\gamma \in \Gamma} \Gamma(n+1, \gamma)$. Notice that $\Gamma(n, \Gamma)$ is contained in, but may not be equal to, the collection of curves in $F^{-n}(\Gamma)$ homotopic rel \mathcal{P}_F to curves in Γ . Define $\kappa_n(\gamma) = \#\Gamma(n, \gamma)$ for each $\gamma \in \Gamma$.

Definition 2. We say that a multicurve Γ is a **Cantor multicurve** if it is pre-stable and $\kappa_n(\gamma) \rightarrow \infty$ as $n \rightarrow \infty$ for all $\gamma \in \Gamma$.

A Cantor multicurve Γ_0 induces a stable Cantor multicurve by the following: Let $\tilde{\Gamma}_n$ be the collection of non-peripheral curves in $F^{-n}(\Gamma_0)$ for $n \geq 1$. Let Γ_n be a subset of $\tilde{\Gamma}_n$ such that no two curves in Γ_n are homotopic rel \mathcal{P}_F and any curve in $\tilde{\Gamma}_n$ is homotopic rel \mathcal{P}_F to a curve in Γ_n . Then Γ_n is a Cantor multicurve and each curve in Γ_n is homotopic to a curve in Γ_{n+1} for $n \geq 1$. Thus $\#\Gamma_n \leq \#\Gamma_{n+1}$. Since for any multicurve Γ , $\#\Gamma \leq \#\mathcal{P}_F - 3$, there is an integer $N \geq 0$ such that $\#\Gamma_N = \#\Gamma_{N+1}$. Thus Γ_N is a stable Cantor multicurve.

Lemma 2.1. Suppose that Γ is an irreducible multicurve. The following statements are equivalent:

- (1) $\#\Gamma(1, \Gamma) > \#\Gamma$.
- (2) $\kappa_1(\gamma) \geq 2$ for some $\gamma \in \Gamma$.
- (3) $\kappa_n(\gamma) \rightarrow \infty$ for some $\gamma \in \Gamma$.
- (4) $\kappa_n(\gamma) \rightarrow \infty$ for all $\gamma \in \Gamma$, i.e., Γ is a Cantor multicurve.
- (5) There is a curve $\beta \in \Gamma$ such that $F^{-1}(\beta)$ has at least two distinct curves contained in $\Gamma(1, \Gamma)$.

Proof. (1) \Leftrightarrow (2): Since Γ is pre-stable, $\Gamma(1, \gamma)$ is non-empty for each $\gamma \in \Gamma$. Thus $\#\Gamma(1, \Gamma) > \#\Gamma$ if and only if $\kappa_1(\gamma) \geq 2$ for some $\gamma \in \Gamma$.

(1) \Leftrightarrow (3): Since Γ is irreducible, $F^{-1}(\gamma)$ has at least one curve contained in $\Gamma(1, \Gamma)$ for each $\gamma \in \Gamma$. Thus if $\#\Gamma(1, \Gamma) > \#\Gamma$, then $\#\Gamma(n+1, \Gamma) > \#\Gamma(n, \Gamma)$ for all $n \geq 1$. So $\#\Gamma(n, \Gamma) \rightarrow \infty$ as $n \rightarrow \infty$. Therefore $\kappa_n(\gamma) \rightarrow \infty$ for some $\gamma \in \Gamma$. Conversely, if $\#\Gamma(1, \Gamma) = \#\Gamma$, then $\#\Gamma(n+1, \Gamma) = \#\Gamma(n, \Gamma)$ for all $n \geq 1$. Therefore $\kappa_n(\gamma) = 1$ for all $\gamma \in \Gamma$ and $n \geq 1$.

(3) \Leftrightarrow (4): Since Γ is irreducible, for each pair $(\gamma, \beta) \in \Gamma \times \Gamma$, there is an integer $n \geq 1$ such that $F^{-n}(\beta)$ has a component δ homotopic to γ rel \mathcal{P}_F and $F^k(\delta)$ is homotopic to a curve in Γ for $1 \leq k < n$. Therefore $\delta \in \Gamma(k, \gamma)$ and hence $\kappa_{n+k}(\gamma) \geq \kappa_n(\beta)$. So $\kappa_n(\gamma) \rightarrow \infty$ if $\kappa_n(\beta) \rightarrow \infty$.

(1) \Leftrightarrow (5): Since Γ is irreducible, $F^{-1}(\gamma)$ has at least one curve contained in $\Gamma(1, \Gamma)$ for each $\gamma \in \Gamma$. Therefore $\#\Gamma(1, \Gamma) > \#\Gamma$ if and only if there is a curve $\beta \in \Gamma$ such that $F^{-1}(\beta)$ has at least two distinct curves contained in $\Gamma(1, \Gamma)$. \square

3 Annular systems

Definition 3. We say that a non-empty open set $\mathcal{A} \subset \hat{\mathbb{C}}$ is a **multi-annulus** if it is a finite union of disjoint open annuli.

Let $\mathcal{A} \subset \hat{\mathbb{C}}$ be a multi-annulus and \mathcal{A}^1 be a multi-annulus essentially contained in \mathcal{A} (i.e. each component U of \mathcal{A}^1 is contained in \mathcal{A} and separates the boundary $\partial\mathcal{A}$). We say that a map $g : \mathcal{A}^1 \rightarrow \mathcal{A}$ is an **annular system** if it is a holomorphic covering and there is an integer $n \geq 1$ such that for each component A of \mathcal{A} , the set $g^{-n}(\mathcal{A}) \cap A$ has at least two connected components (in particular it is non-empty). Its **Julia set** is defined as $\mathcal{J}_g := \bigcap_{n \geq 0} g^{-n}(\mathcal{A})$.

We say that an annular system $g : \mathcal{A}^1 \rightarrow \mathcal{A}$ is **proper** if \mathcal{A}^1 is compactly contained in \mathcal{A} (denoted by $\mathcal{A}^1 \subset\subset \mathcal{A}$); or **exact** if for every component A of \mathcal{A} , every component of ∂A is also a component of $\partial(\mathcal{A} \cap \mathcal{A}^1)$.

Remark. Here we say that a map $g : \mathcal{A}^1 \rightarrow \mathcal{A}$ is a holomorphic covering if for each component U of \mathcal{A}^1 , the set $g(U)$ is a component of \mathcal{A} and $g : U \rightarrow g(U)$ is a holomorphic covering. We do not require that g is surjective.

To give an example of an exact annular system, one may take $\mathcal{A} = \{e^0 < |z| < e^1\}$, $\mathcal{A}^1 = \{e^0 < |z| < e^{\frac{1}{3}}\} \cup \{e^{\frac{2}{3}} < |z| < e^1\}$, and

$$g : \mathcal{A}^1 \rightarrow \mathcal{A}, \quad z \mapsto \begin{cases} z^3 & \text{if } e^0 < |z| < e^{\frac{1}{3}} \\ e^3 z^{-3} & \text{if } e^{\frac{2}{3}} < |z| < e^1. \end{cases}$$

Proposition 3.1. *Let $g : \mathcal{A}^1 \rightarrow \mathcal{A}$ be an annular system. Then there is an integer $N \geq 1$ such that $\deg g^N|_U \geq 2$ for any component U of $g^{-N}(\mathcal{A})$.*

Proof. Let m be the number of components of \mathcal{A}^1 . Assume that there is a component U of $g^{-(m+2)}(\mathcal{A})$ such that $\deg g^{m+2}|_U = 1$. Then there exist integers $0 \leq k_1 < k_2 \leq m+1$ such that both $g^{k_1}(U)$ and $g^{k_2}(U)$ are contained in the same component of \mathcal{A}^1 , denote it by B . Let $C \subset B$ be the component of $g^{-(k_2-k_1)}(B)$ containing $g^{k_1}(U)$, then

$$\deg(g^{k_2-k_1} : C \rightarrow B) = \deg(g^{k_2-k_1} : g^{k_1}(U) \rightarrow g^{k_2}(U)) = 1.$$

Thus $\text{mod } C = \text{mod } B$ and hence $C = B$. So $B \cap g^{-(k_2-k_1)}(\mathcal{A}) = B$. It follows that $B \cap g^{-n(k_2-k_1)}(\mathcal{A}) = B$ for all n . As $g^{-n}(\mathcal{A})$ is a decreasing sequence with respect to n , we conclude that $B \cap g^{-n}(\mathcal{A}) = B$ for all n . This contradicts the assumption that $B \cap g^{-n(k_2-k_1)}(\mathcal{A})$ should be disconnected for some n .

So $\deg g^{m+2}|_U \geq 2$ for any component U of $g^{-(m+2)}(\mathcal{A})$. \square

Let $g : \mathcal{A}^1 \rightarrow \mathcal{A}$ be an annular system. Denote $\mathcal{A}^n = g^{-n}(\mathcal{A})$ for $n > 1$. For $n \geq 1$ and any connected component K of \mathcal{J}_g , denote by $A^n(K)$ the component of \mathcal{A}^n containing K . Then $K \subset \bigcap_{n \geq 1} A^n(K)$. We will say that K is **periodic** if there is an integer $p \geq 1$ such that $g^p(A^n(K)) = A^{n-p}(K)$ for all $n > p$; **pre-periodic** if $f^k(K)$ is periodic for some integer $k \geq 1$; or **wandering** otherwise.

Proposition 3.2. *Let $g : \mathcal{A}^1 \rightarrow \mathcal{A}$ be an exact annular system. Let $\{A^n\}$ be a nested sequence of annuli of $\{g^{-n}(\mathcal{A})\}$, i.e. for every n the set A^n is a component of $g^{-n}(\mathcal{A})$, and $A^{n+1} \subset A^n$. Then either $\bigcap_{n \geq 0} A^n = \emptyset$ or for every $n \geq 0$, there is an integer $m > n$ such that $A^m \subset \subset A^n$.*

On the other hand, any component K of \mathcal{J}_g is a continuum and $K = \bigcap_{n \geq 1} A^n(K)$. If K is (pre)-periodic, then it is a quasicircle.

Proof. Consider the nested sequence $\{A^n\}$. Either there is an integer $N \geq 0$ such that A^N shares a common boundary component with A^n for every $n \geq N$, or for any $n \geq 0$ there is an integer $m > n$ such that $A^m \subset \subset A^n$.

Assume that we are in the former case. Since \mathcal{A} has only finitely many components, there are integers $k > j > i > N$ such that $g^i(A^i) = g^j(A^j) = g^k(A^k)$. Set $p = j - i$, $B^n = g^i(A^{n+i})$ for $n \geq 0$ and $C^n = g^p(B^{n+p})$ for $n \geq 0$. Then $B^0 = C^0 = g^i(A^i)$ and $\{B^n\}$ (resp. $\{C^n\}$) is a nested sequence of annuli of $g^{-n}(\mathcal{A})$ which share a common boundary component with B^0 . We have $g^p(B^p) = g^i(A^i) = B^0$.

Note that $B^p \neq B^0$. Otherwise g^p is a conformal map on B^p and hence $B^n = B^0$ for all $n \geq 1$. It contradicts the assumption that $B^0 \cap g^{-n}(\mathcal{A})$ is disconnected for some $n \geq 1$. Therefore $B^p \subset \subset B^0$.

Assume at first that the common boundary component of C^n with B^0 is equal to that of B^n with B^0 . Denote this boundary component by L . Now both C^n

and B^n are components of $g^{-n}(\mathcal{A})$ contained in B^0 and sharing L as a boundary component. By the uniqueness of such components, we have $C^n = B^n$ for all $n \geq 0$. Then $g^p(B^{n+p}) = C^n = B^n$ for $n \geq 1$. It follows that $g^p(z) \rightarrow L$ as $z \rightarrow L$.

Let U be the component of $\widehat{\mathbb{C}} \setminus L$ containing B^0 and $\phi : U \rightarrow \mathbb{D}$ be a conformal map to the unit disk. Then $h := \phi \circ g^p \circ \phi^{-1}$ is a holomorphic covering from $\phi(B^p)$ to $\phi(B^0)$, with $|\phi(z)| \rightarrow 1$ as $|z| \rightarrow 1$. By the reflection principle, the map h can be extended to a holomorphic covering from the annulus V_1 to V , where V_1 (or V) is the union of $\phi(B^p)$ (or $\phi(B^0)$) with its reflection and the unit circle, respectively. Since $V_1 \subset \subset V$, the map h is expanding with respect to the hyperbolic metric of V . So $\bigcap_{n>0} h^{-n}(V) = \partial\mathbb{D}$ and hence $\bigcap_{n>0} h^{-n}(\phi(B^0)) = \emptyset$. Note that $\phi(B^{np}) = h^{-n}(\phi(B^0))$. Therefore $\bigcap_{n>0} B^{np} = \emptyset$ and hence $\bigcap_{n>0} A^n = \emptyset$.

Assume now the common boundary component of C^n with B^0 is not equal to that of B^n with B^0 . But B^0 has only two boundary components. So one, denoted by L_C , is shared with C^n and the other, denoted by L_B , is shared with B^n . Set now $q = k - j$ and define $D^n = g^q(C^{n+q})$ for $n \geq 0$. Then $D^0 = B^0 = C^0$ and the D^n 's share a common boundary component with B^0 . It must be either L_C or L_B . If it is L_C , repeat the above argument with p replaced by q . Or else, repeat the above argument but with p replaced by $p + q$. The rest follows.

Now suppose that K is a component of \mathcal{J}_g . Then for any $n \geq 0$, there is an integer $m > n$ such that $A^m(K) \subset \subset A^n(K)$ by the above argument. Thus $\bigcap_{n \geq 1} A^n(K)$ is a continuum containing K and contained in \mathcal{J}_g , hence is equal to K .

Let K be a periodic component of \mathcal{J}_g of period $p \geq 1$. Then $g^p(A^{n+p}(K)) = A^n(K)$ for $n > 0$. If $A^{kp+1}(K)$ for $k = 0, 1, 2$ share a common boundary component, then all $A^n(K)$ for $n \geq 1$ share a common boundary component. Thus $\bigcap_{n \geq 1} A^n(K) = \emptyset$ by the above argument. This is not possible. So $A^{2p+1}(K) \subset \subset A^1(K)$. Now applying quasiconformal surgery, we have a quasiconformal map ϕ of $\widehat{\mathbb{C}}$ such that $\phi \circ g^{2p} \circ \phi^{-1} = z^d$ in a neighborhood of K , where $|d| = \deg(g^{2p}|_{A^{2p+1}(K)}) \geq 2$. Thus K is a quasicircle.

It follows that every preperiodic component of \mathcal{J}_g is also a quasicircle. \square

Corollary 3.3. *Let $g : \mathcal{A}^1 \rightarrow \mathcal{A}$ be an exact annular system. Let $E \subset \mathcal{A}$ be a multicurve containing exactly one essential Jordan curve in each component of \mathcal{A} . Let A be a component of \mathcal{A} . Set*

$$A(n, E) = \begin{cases} \emptyset & \text{if } g^{-n}(E) \cap A = \emptyset, \\ g^{-n}(E) \cap A & \text{if } g^{-n}(E) \cap A \text{ consists of a single curve,} \\ \text{or else the largest closed annulus bounded by two curves in } g^{-n}(E) \cap A. \end{cases}$$

Then $g^{-n}(E) \cap A \subset A(n, E) \subset \subset A$. Furthermore, for any compact set $G \subset A$, we have $G \subset A(n, E)$ for large enough n .

Proof. Label the two boundary components of A by $\partial_+ A$ and $\partial_- A$. By the definition of an exact annular system, for each $n \geq 1$ there are two annuli (possibly equal) A_+^n, A_-^n of $g^{-n}(\mathcal{A})$ contained in A such that $\partial_+ A$ (resp. $\partial_- A$) is also a boundary component of A_+^n (resp. A_-^n). We have $\bigcap_{n \geq 1} (A_+^n \cup A_-^n) = \emptyset$ by Proposition 3.2. Therefore for any compact subset G of A , we have $G \cap (A_+^n \cup A_-^n) = \emptyset$ for all large enough n . But each of A_+^n and A_-^n contains a curve in $g^{-n}(E)$ and the two curves bound $A(n, E)$. So $G \subset A(n, E)$ when n is large enough. \square

The dynamics of an exact annular system $g : \mathcal{A}^1 \rightarrow \mathcal{A}$ can be characterized by a linear system as the following. Denote $\mathcal{A} = A_1 \cup \cdots \cup A_n$ and $\mathcal{A}^1 = A_1^1 \cup \cdots \cup A_m^1$ the disjoint unions of annuli. Let $\mathcal{I} = I_1 \cup \cdots \cup I_n$ and $\mathcal{I}^1 = I_1^1 \cup \cdots \cup I_m^1$ be unions of disjoint closed intervals on \mathbb{R}^1 such that

- Each I_j has unit length,
- $I_i^1 \subset I_j \iff A_i^1 \subset A_j$,
- $\partial \mathcal{I} \subset \partial \mathcal{I}^1$.

Define a map $\sigma : \mathcal{I}^1 \rightarrow \mathcal{I}$ by

- $\sigma(I_i^1) = I_j \iff g(A_i^1) = A_j$,
- σ is affine on each I_i^1 .

Denote $\mathcal{I}^n := \sigma^{-n}(\mathcal{I})$, $\mathcal{J}_\sigma := \bigcap_{n \geq 1} \mathcal{I}^n$ and $\mathcal{B}_\sigma = \bigcup_{n \geq 1} \partial \mathcal{I}^n$ the end point set. Then $\mathcal{B}_\sigma \subset \mathcal{J}_\sigma$. We will say that the dynamics of σ is **expanding** if there are constants $\lambda > 1$ and $C > 0$ such that for any $n \geq 1$ and any point $x \in \mathcal{I}^n$, $|(\sigma^n)'(x)| \geq C\lambda^n$.

Proposition 3.4. *The dynamics of σ is expanding and \mathcal{J}_σ is a Cantor set. Moreover, there is a continuous map $\pi : \mathcal{J}_g \rightarrow \mathcal{J}_\sigma$ such that*

- (1) $\pi(\mathcal{J}_g) = \mathcal{J}_\sigma \setminus \mathcal{B}_\sigma$,
- (2) $\pi^{-1}(x)$ is a component of \mathcal{J}_g for each $x \in \mathcal{J}_\sigma \setminus \mathcal{B}_\sigma$, and
- (3) $\sigma \circ \pi = \pi \circ g$ on \mathcal{J}_g .

Proof. To prove that the dynamics of σ is expanding, we only need to show that there is an integer $n \geq 1$ such that for any $x \in \mathcal{I}^n$, $|(\sigma^n)'(x)| > 1$. Set

$$l = \min_{1 \leq i \leq n} |I_i| \text{ and } l_k = \max_{I^k \subset \mathcal{I}^k} |I^k|,$$

where $|\cdot|$ denotes the length of the interval. Then $l_{k+1} \leq l_k$ for any $k \geq 1$. To prove $|(\sigma^n)'| > 1$, it is sufficient to show that there is an integer $n \geq 1$ such that $l_n < l$. Actually, we will prove $\lim_{k \rightarrow \infty} l_k = 0$ as follows. Assume $\lim_{k \rightarrow \infty} l_k = l_\infty > 0$. Then for each $k \geq 1$, there is a component of \mathcal{I}^k whose length is at least l_∞ . Therefore, there exists a sequence $\{I^k\}_{k \geq 1}$ with I^k a component of \mathcal{I}^k , such that $I^k \supset I^{k+1}$ and $|I^k| \geq l_\infty$. Set $I^\infty = \bigcap_k I^k$. Then I^∞ is a component of \mathcal{J}_σ . It can not be wandering since the total length of \mathcal{I} is finite and $|\sigma'| \geq 1$ (the interval-wise affine map σ sends intervals of at most unit length onto intervals of unit length). Therefore there exist integers $k \geq 0$ and $p \geq 1$ such that $\sigma^k(I^\infty)$ is p -periodic. Let I_i^1 be the component of \mathcal{I}^1 with $\sigma^k(I^\infty) \subset I_i^1$. Then $\sigma^{-p}(I_i^1)$ has a component I^{p+1} contained in I_i^1 and $|(\sigma^p)'| = 1$ on I^{p+1} . Thus $I^{p+1} = I_i^1$. This contradicts the condition that $I_i^1 \cap \mathcal{I}^n$ is disconnected for some $n \geq 2$ since g is an annular system.

Now each component of \mathcal{J}_σ is a single point since the dynamics of σ is expanding. For any $x \in \mathcal{J}_\sigma$, let $I^k(x)$ be the component of \mathcal{I}^k containing x . If $I^k(x)$ have common endpoints for k large enough, then the other endpoints of $I^k(x)$ converge to x . Otherwise, any endpoints of $I^k(x)$ converges to x . So \mathcal{J}_σ is a perfect set hence a Cantor set.

For any point $z \in \mathcal{J}_g$, the itinerary of z is defined by $i(z) = (i_0, i_1, \dots)$ if $g^k(z) \in A_{i_k}^1$. For any point $x \in \mathcal{J}_\sigma$, the itinerary of x is defined by $i_*(x) = (j_0, j_1, \dots)$ if $\sigma^k(x) \in I_{j_k}^1$. Note that different points in \mathcal{J}_σ have different itineraries (this is why we use \mathcal{A}^1 instead of \mathcal{A} to define itineraries).

We claim that for each $x \in \mathcal{B}_\sigma$, there is no other point in \mathcal{J}_g sharing the same itinerary with x . To prove this we just need to apply Proposition 3.2 and the definition of σ .

Define a map $\pi : \mathcal{J}_g \rightarrow \mathcal{J}_\sigma$ by $\pi(z) = x$ if $i(z) = i_*(x)$. Then $\pi(\mathcal{J}_g) = \mathcal{J}_\sigma \setminus \mathcal{B}_\sigma$ and part (3) of the proposition holds automatically. Fix any point $x \in \mathcal{J}_\sigma \setminus \mathcal{B}_\sigma$. The set $\pi^{-1}(x)$ is a component of \mathcal{J}_g and the sequence $\{I^k(x) \cap \mathcal{J}_\sigma\}$ forms a basis of neighborhoods of x in \mathcal{J}_σ . Now $\pi^{-1}(\{I^k(x) \cap \mathcal{J}_\sigma\}) = A^k(\pi^{-1}(x)) \cap \mathcal{J}_g$ is open in \mathcal{J}_g for every k . So π is continuous. \square

Since \mathcal{B}_σ is a countable set and the set of pre-periodic points is also countable, we have:

Corollary 3.5. *There are uncountably many wandering components in \mathcal{J}_g .*

Theorem 3.6. *Let $g : \mathcal{A}^1 \rightarrow \mathcal{A}$ be an exact annular system and K be a component of \mathcal{J}_g . Let U and V be the two components of $\widehat{\mathbb{C}} \setminus K$. Then $\partial U = \partial V = K$.*

Proof. Assume that each component of \mathcal{A} contains at least two components of \mathcal{A}^1 (otherwise we consider g^n for some $n \geq 1$ by the definition). Then $\|g'\| > 1$ under the hyperbolic metric of \mathcal{A} .

Suppose that K is wandering (otherwise K is a quasicircle by Proposition 3.2 and hence the theorem holds). Then there is a component L of \mathcal{J}_g such that $\pi(L)$ is contained in the ω -limit set of $\pi(K)$, where π is the map defined in Proposition 3.4. This means that for any $m \geq 0$, there are infinitely many components in the forward orbit of K passing through $A^m(L)$, where $A^m(L)$ is the component of $g^{-m}(\mathcal{A})$ containing L .

When m is large enough, $A^m(L) \subset\subset A^0(L)$ by Proposition 3.2. Denote by $\{n_k \geq 1\}$ the increasing sequence such that $g^{n_k}(K) \subset A^m(L)$. Then $g^{n_k}(A^{m+n_k}(K)) = A^m(L)$.

Pick a closed annulus W bounded by smooth curves such that $W \subset A^0(L)$ and $A^m(L) \subset W$. Then there exists a constant $\lambda > 1$ such that $\|g'(u)\| \geq \lambda > 1$ for every $u \in g^{-1}(W)$.

Denote by W_k the component of $g^{-n_k}(W)$ that contains K . Then $\|(g^{n_k})'(z)\| \geq \lambda^k$ for every $z \in W_k$ since the portion of the z -orbit $z, g(z), \dots, g^{n_k-1}(z)$ passes through k times the set $g^{-1}(W)$ and for the remaining times $\|g'\| > 1$.

For each closed annulus $W' \subset\subset \mathcal{A}$ with smooth boundary, define

$$\omega(W') = \sup_{z \in W'} \{d_{W'}(z, \partial_+ W') + d_{W'}(z, \partial_- W')\},$$

where $\partial_\pm W'$ denotes the two boundary components of W' and $d_{W'}(z, E)$ denotes the infimum of the hyperbolic length of arcs connecting z to E within W' . Then $\omega(W_k) \leq \lambda^{-k} \omega(W)$ and hence $\omega(W_k) \rightarrow 0$ as $k \rightarrow \infty$.

Clearly $\partial U \cup \partial V \subset K$. In order to prove $\partial U = \partial V = K$ we only need to show $K \subset \partial U$ (by symmetry). Otherwise, assume $z \in K \setminus \partial U$. Label the boundary components of W_k by $\partial_\pm W_k$ so that $\partial_+ W_k \subset U$. Then

$$\omega(W_k) \geq d(z, \partial_+ W_k) \geq d(z, \partial U) > 0.$$

This contradicts the fact $\omega(W_k) \rightarrow 0$ as $k \rightarrow \infty$. \square

Note that the above proof does not claim that a wandering component of \mathcal{J}_g is a Jordan curve. What's missing is the local connectivity. Actually X. Buff constructed an example of exact annular systems whose Julia set has a non-locally connected wandering component.

The next theorem gives a sufficient condition about the local connectivity of wandering components. The idea of the proof comes from [21].

Theorem 3.7. *Let $g : \mathcal{A}^1 \rightarrow \mathcal{A}$ be an exact annular system. Suppose that g is expanding, i.e. there is a metric ρ on \mathcal{A} and a constant $\lambda > 1$ such that $\|g'\| \geq \lambda$ with respect to ρ . Then every component of \mathcal{J}_g is a Jordan curve.*

Proof. Pick a pre-periodic component of \mathcal{J}_g in each component of \mathcal{A} and denote by Γ_0 the collection of them. It is a multicurve. Denote by Γ_n the collection of curves in $g^{-n}(\Gamma_0)$. Then each curve in Γ_n is a quasicircle, and is disjoint from any curve in Γ_m , $m \neq n$.

For each curve $\beta \in \Gamma_1$, there is a unique curve $\gamma \in \Gamma_0$ such that β and γ are contained in the same component of \mathcal{A} . If $\beta \neq \gamma$, there is an isotopy $\Phi_\beta : S^1 \times [0, 1] \rightarrow \mathcal{A}$ from γ to β such that $\phi_t := \Phi_\beta(\cdot, t)$ is a homeomorphism for any $t \in [0, 1]$, and in particular, $\phi_0(S^1) = \gamma$, $\phi_1(S^1) = \beta$ and $\phi_t(S^1)$ is a curve between β and γ . If $\beta = \gamma$, define $\Phi_\beta(\cdot, t) : S^1 \rightarrow \beta$ to be a homeomorphism independent on t .

Define the homotopic length of a path $\delta : [0, 1] \rightarrow \mathcal{A}$ by

$$\text{h-length}(\delta) = \inf\{\text{length of } \zeta \text{ with metric } \rho\},$$

where the infimum is taken over all the path ζ from $\delta(0)$ to $\delta(1)$ and homotopic to δ . Then $\text{h-length}(\tilde{\delta}) \leq \frac{1}{\lambda} \text{h-length}(\delta)$ for any lift $\tilde{\delta}$ of δ under the map g since $\|g'\| \geq \lambda$.

For each $\beta \in \Gamma_1$ and any $s \in S^1$, $\Phi_\beta(s, \cdot)$ maps the interval $[0, 1]$ to a path $\delta_{\gamma,s}$ in the closed annulus $\Phi_\beta(S^1 \times [0, 1])$ which connects two points in each of its boundary. So there is a constant $C > 0$ such that $\text{h-length}(\delta_{\gamma,s}) < C$ for each $\gamma \in \Gamma_1$ and any $s \in S^1$.

For each wandering component K of \mathcal{J}_g , let α_n be the unique curve of Γ_n with $\alpha_n \subset A^n(K)$. Then $g^n(\alpha_n) \in \Gamma_0$ and $\beta := g^n(\alpha_{n+1}) \in \Gamma_1$ are contained in the same component of \mathcal{A} . Now the isotopy Φ_β from $g^n(\alpha_n)$ to β defined above can be lifted to an isotopy from α_n to α_{n+1} , denote it by $\Psi_n : S^1 \times [0, 1] \rightarrow A^n(K)$, by the following commutative diagram:

$$\begin{array}{ccc} S^1 \times [0, 1] & \xrightarrow{\Psi_n} & A^n(K) \\ P_d \downarrow & & \downarrow g^n \\ S^1 \times [0, 1] & \xrightarrow{\Phi_\beta} & g^n(A^n(K)), \end{array}$$

where $d = \deg(g|_{A^n(K)})$ and $P_d(s, t) = (s^d, t)$, i.e. $P(\cdot, t)$ is a covering of S^1 with degree d . Set $\psi_{n,t} := \Psi_n(\cdot, t)$. It is a homeomorphism for any $t \in [0, 1]$, and in particular, $\psi_{n,0}(S^1) = \alpha_n$ and $\psi_{n,1}(S^1) = \alpha_{n+1}$. For any $s \in S^1$, $\Psi_n(s, t)(S^1)$ is a path connecting a point in α_n with a point in α_{n+1} whose homotopic length is less than $C\lambda^{-n}$.

These isotopies Ψ_n can be pasted together to a continuous map $\Psi : S^1 \times [0, \infty) \rightarrow \mathcal{A}$ as the following:

$$\Psi(s, t) = \begin{cases} \Psi_0(s, t) & \text{on } S^1 \times [0, 1] \\ \Psi_1(\psi_{1,0}^{-1} \circ \psi_{0,1}(s), t - 1) & \text{on } S^1 \times [1, 2] \\ \vdots & \vdots \\ \Psi_n(\psi_{n,0}^{-1} \circ \psi_{n-1,1} \circ \cdots \circ \psi_{1,0}^{-1} \circ \psi_{0,1}(s), t - n) & \text{on } S^1 \times [n, n+1] \\ \vdots & \vdots \end{cases}$$

Set $h_t = \Psi(\cdot, t)$. Then $h_n(S^1) = \alpha_n$. For each $s \in S^1$ and any integers $m > n \geq 0$, the homotopic length of the path $\zeta_s(n, m) := \{\Psi(s, t) : n \leq t \leq m\}$ satisfies:

$$\text{h-length } (\zeta_s(n, m)) \leq C\lambda^{-n} + \cdots + C\lambda^{1-m} \leq \frac{C}{(\lambda - 1)\lambda^{n-1}}.$$

Note that the two endpoints of $\zeta_s(n, m)$ are $h_n(s) \in \alpha_n$ and $h_m(s) \in \alpha_m$. The above inequality shows that $\{h_n\}$ is a Cauchy sequence and hence converges uniformly to a continuous map h . Since $\alpha_n \subset A^n(K)$, we have $h(S^1) = K$ by Theorem 3.6. Therefore K is locally connected and hence is a Jordan curve (see [21], Lemma 5.1). \square

4 From multicurves to annular systems

Let f be a rational map. Denote by \mathcal{J}_f the Julia set of f and \mathcal{F}_f the Fatou set of f . Refer to [2, 5, 18, 19], for definitions and basic properties. In this section, we shall prove Theorem 4.1 which is a more precise version of Theorem 1.1.

We say that an annulus $A \subset \widehat{\mathbb{C}} \setminus \mathcal{P}_f$ is homotopic rel \mathcal{P}_f to a Jordan curve γ (or an annulus A') in $\widehat{\mathbb{C}} \setminus \mathcal{P}_f$ if essential Jordan curves in A are homotopic to γ (or essential curves in A') rel \mathcal{P}_f ; and a multi-annulus \mathcal{A} is homotopic rel \mathcal{P}_f to a multicurve Γ (or a multi-annulus \mathcal{A}') if each component of \mathcal{A} is homotopic to a curve in Γ (or a component of \mathcal{A}') rel \mathcal{P}_f and each curve in Γ (or each component of \mathcal{A}') is homotopic to a component of \mathcal{A} .

Theorem 4.1. *Let f be a postcritically finite rational map with a Cantor multicurve Γ . Then there is unique a multi-annulus $\mathcal{A} \subset \widehat{\mathbb{C}} \setminus \mathcal{P}_f$ homotopic rel \mathcal{P}_f to Γ such that $f : \mathcal{A}^1 \rightarrow \mathcal{A}$ is an exact annular system, where \mathcal{A}^1 is the union of components of $f^{-1}(\mathcal{A})$ that are homotopic rel \mathcal{P}_f to curves in Γ .*

In order to prove this theorem, we will first modify the rational map to a Thurston-equivalent branched covering having a topological exact annular system. Instead of using the deep characterization result of Thurston, we will then apply a theorem due to Rees and Shishikura to obtain a semi-conjugacy from the branched covering to the rational map.

Definition 4. *Two postcritically finite branched coverings f and F of $\widehat{\mathbb{C}}$ are said to be **Thurston-equivalent** if there exists a pair of orientation-preserving homeomorphisms (ϕ_0, ϕ_1) of $\widehat{\mathbb{C}}$ such that ϕ_1 is homotopic to ϕ_0 rel \mathcal{P}_f and $F \circ \phi_1 = \phi_0 \circ f$.*

Let f be a postcritically finite rational map with a Cantor multicurve Γ . Then there exists a multi-annulus $\mathfrak{C} \subset \widehat{\mathbb{C}} \setminus \mathcal{P}_f$ homotopic to Γ rel \mathcal{P}_f such that its boundary $\partial\mathfrak{C}$ is a disjoint union of Jordan curves in $\widehat{\mathbb{C}} \setminus \mathcal{P}_f$. Let \mathfrak{C}^* be the union of all the components of $f^{-1}(\mathfrak{C})$ which are homotopic to curves in Γ . Then for each $\gamma \in \Gamma$, there is at least one component of \mathfrak{C}^* homotopic to γ rel \mathcal{P}_f since Γ is pre-stable.

For each $\gamma \in \Gamma$, denote by $\mathfrak{C}^*(\gamma)$ the smallest annulus containing all the components of \mathfrak{C}^* which are homotopic to γ rel \mathcal{P}_f . Then its boundary are two Jordan curves in $\widehat{\mathbb{C}} \setminus \mathcal{P}_f$ homotopic to γ rel \mathcal{P}_f . Set $\mathfrak{C}^*(\Gamma) = \bigcup_{\gamma \in \Gamma} \mathfrak{C}^*(\gamma)$. Then there exist a neighborhood U of \mathcal{P}_f and a homeomorphism θ_0 of $\widehat{\mathbb{C}}$ such that θ_0 is homotopic to the identity rel $\mathcal{P}_f \cup U$ and $\theta_0(\mathfrak{C}) = \mathfrak{C}^*(\Gamma)$. Set $F := f \circ \theta_0$ and $\mathfrak{C}^1 := \theta_0^{-1}(\mathfrak{C}^*)$, then $\mathcal{P}_F = \mathcal{P}_f$ and F is Thurston-equivalent to f via the pair (θ_0, id) . Moreover, the restriction $F|_{\mathfrak{C}^1} : \mathfrak{C}^1 \rightarrow \mathfrak{C}$ is a topological exact annular system.

By Rees-Shishikura's semi-conjugacy result, stated in the appendix as Theorem A.1, there exist a neighborhood V of \mathcal{P}_f and a sequence $\{\phi_n\}$ ($n \geq 1$) of homeomorphisms of $\widehat{\mathbb{C}}$ homotopic to the identity rel $\mathcal{P}_F \cup V$ such that $f \circ \phi_n = \phi_{n-1} \circ F$. Moreover, the sequence $\{\phi_n\}$ converges uniformly to a continuous onto map h of $\widehat{\mathbb{C}}$ and $f \circ h = h \circ F$. Define

$$T = \{w \in \widehat{\mathbb{C}} : h^{-1}(w) \text{ crosses some component of } \mathfrak{C}\},$$

here we say a continuum E **crosses** an annulus C if E intersects both boundary components of C . Then $T \subset \mathcal{J}_f$ by Theorem A.1 (3). It is easy to see that T is closed.

Lemma 4.2. *The set T is empty.*

This lemma plays a crucial role in the proof of Theorem 4.1. Here the property of Cantor multicurves is essential. The next topological lemma will be used in its proof.

Lemma 4.3. *Let $\Gamma = \gamma_1 \cup \dots \cup \gamma_n$ be a union of finitely many pairwise disjoint real-analytic Jordan curves on $\widehat{\mathbb{C}}$ and $L \subset \widehat{\mathbb{C}}$ be a continuum. Then for any Jordan domain Δ containing L , there is an integer $N \geq 0$ such that for any two distinct points $z_1, z_2 \in L$, there is a Jordan arc β in Δ connecting z_1 with z_2 such that $\#(\beta \cap \Gamma) \leq N$.*

Proof. Reduce Δ to a sub-Jordan-domain Δ^* with a real-analytic boundary so that we still have $L \subset \Delta^*$. Then $\Gamma \cap \Delta^*$ consists of finitely many, say N , open sub-arcs of Γ with ends on $\partial\Delta^*$. These arcs cut Δ^* into finitely many Jordan domains with piece-wise real-analytic boundaries. There is therefore a Jordan arc $\beta \subset \Delta^* \subset \Delta$ connecting z_1 to z_2 and intersecting Γ at most N times. \square

Proof of Lemma 4.2. Assume $T \neq \emptyset$ by contradiction. Then $f(T) \subset T$. In fact, suppose $w \in T$, i.e., $h^{-1}(w)$ crosses some component of \mathfrak{C} , then $h^{-1}(w)$ crosses some component C^1 of \mathfrak{C}^1 . By Theorem A.1 (6), $h^{-1}(f(w)) = F(h^{-1}(w))$. So $h^{-1}(f(w))$ crosses $F(C^1)$ which is a component of \mathfrak{C} , so $f(w) \in T$. Set $T_\infty = \bigcap_{n \geq 0} f^n(T)$. Then T_∞ is a non-empty closed set and $f(T_\infty) = T_\infty$.

Pick one point $w_0 \in T_\infty$. Since $f(T_\infty) = T_\infty$, there exists a sequence of points $\{w_n\}_{n \geq 0}$ in T_∞ such that $f(w_{n+1}) = w_n$ (i.e. we have the tail of a backward orbit). Either w_n is periodic for all $n \geq 0$ or there is an integer $n_0 \geq 0$ such that w_n is not periodic for all $n \geq n_0$. In the former case no w_n can be a critical point of f since $w_n \in \mathcal{J}_f$. In the latter case, there exists an integer $n_1 \geq 0$ such that w_n for $n \geq n_1$ are non-critical points of f . So in both cases, we have a sequence of points $\{w_n\}_{n \geq n_1}$ in $T_\infty \setminus \Omega_f$ such that $f(w_{n+1}) = w_n$.

Set $L_n = h^{-1}(w_n)$. By Theorem A.1 (4), L_n is a component of $F^{-n}(L_0)$ and there exists a topological disk $\Delta_0 \supset L_0$ such that $F^n : \Delta_n \rightarrow \Delta_0$ is a homeomorphism for $n \geq 1$, where Δ_n is the component of $F^{-n}(\Delta_0)$ containing L_n .

Pick an essential real-analytic Jordan curve in each component of \mathfrak{C} . They form a Cantor multicurve Γ_1 . By Lemma 4.3, there exists an integer $N \geq 0$ such that for any two distinct points $z_1, z_2 \in L_0$, there is a Jordan arc $\beta \subset \Delta_0$ connecting z_1 with z_2 , such that $\#(\beta \cap \Gamma_1) \leq N$.

On the other hand, since Γ_1 is a Cantor multicurve, there is an integer $m > 0$ such that for each component C of \mathfrak{C} , there are at least $N+1$ components of $F^{-m}(\mathfrak{C})$ are contained in C essentially. Since L_m crosses a component of \mathfrak{C} , there are two

distinct points $z_1, z_2 \in L_m$ such that $F^{-m}(\Gamma_1)$ has at least $N + 1$ components separating z_1 from z_2 .

Now $F^m(z_1), F^m(z_2) \in L_0$, so there is a Jordan arc $\beta \subset \Delta_0$ connecting $F^m(z_1)$ with $F^m(z_2)$, such that $\#(\beta \cap \Gamma_1) \leq N$. Let δ be the component of $F^{-m}(\beta)$ connecting z_1 and z_2 , then $\#(\delta \cap F^{-m}(\Gamma_1)) \leq N$ since $F^m : \delta \rightarrow \beta$ is a homeomorphism. This contradicts the fact that $F^{-m}(\Gamma_1)$ has at least $N + 1$ components separating z_1 from z_2 . \square

Corollary 4.4. *For any $n \geq 0$ and any distinct components E_1, E_2 of $F^{-n}(\widehat{\mathbb{C}} \setminus \mathfrak{C})$, $h(E_1)$ is disjoint from $h(E_2)$.*

Proof. E_1 and E_2 are separated by a component A of $F^{-n}(\mathfrak{C})$. If $h(E_1) \cap h(E_2) \neq \emptyset$, pick a point $w \in h(E_1) \cap h(E_2)$, then $h^{-1}(w)$ crosses A . So $F^n(h^{-1}(w)) = h^{-1}(f^n(w))$ (by Theorem A.1 (4)) crosses $F^n(A)$. This contradicts Lemma 4.2. \square

Proof of Theorem 4.1. Denote $\mathcal{E} = \widehat{\mathbb{C}} \setminus \mathfrak{C}$. Denote $\widehat{E} = h^{-1}(h(E))$ for any continuum E of $\widehat{\mathbb{C}}$. It is also a continuum by Theorem A.1 (5). If E^1 is a component of $F^{-1}(\mathcal{E})$, then \widehat{E}^1 is a component of $F^{-1}(\widehat{\mathcal{E}})$ by Corollary 4.4 and Theorem A.1 (7).

For each component C of \mathfrak{C} , there are two distinct components E_+, E_- of \mathcal{E} such that $C = A(E_+, E_-)$, where $A(E_+, E_-)$ denotes the unique annular component of $\widehat{\mathbb{C}} \setminus (E_+ \cup E_-)$. Define $\widetilde{C} := A(\widehat{E}_+, \widehat{E}_-)$. It is an annulus essentially contained in C . Moreover, we claim that the following statements hold:

- (a) $h^{-1}(h(\widetilde{C})) = \widetilde{C}$.
- (b) $\widetilde{C} \cap \widehat{E} = \emptyset$ for any $E \subset \widehat{\mathbb{C}}$ with $E \cap \widetilde{C} = \emptyset$.
- (c) $h(\widetilde{C})$ is an annulus homotopic to C rel \mathcal{P}_f .

Proof. (a) For any point $z \in \widetilde{C}$, if $h^{-1}(h(z))$ is not contained in \widetilde{C} , then it must intersect $E_+ \cup E_-$. So $z \in \widehat{E}_+ \cup \widehat{E}_-$, a contradiction.

(b) If $z \in \widetilde{C} \cap \widehat{E}$, then $h^{-1}(h(z)) \subset \widetilde{C}$ and hence is disjoint from E . This contradicts $z \in \widehat{E}$.

(c) Let Q_+, Q_- be the two components of $\widehat{\mathbb{C}} \setminus \widetilde{C}$. Then both \widehat{Q}_+ and \widehat{Q}_- are disjoint from \widetilde{C} by (b). Moreover, they are also disjoint from each other since $h^{-1}(h(z))$ does not cross C for any point $z \in \widehat{\mathbb{C}}$ by Lemma 4.2. So $\widehat{\mathbb{C}} \setminus h(\widetilde{C})$ has exactly two components, $h(Q_+)$ and $h(Q_-)$. Therefore $h(\widetilde{C})$ is an annulus. Since h is homotopic to the identity rel \mathcal{P}_f , the set $h(\widetilde{C})$ is homotopic to C rel \mathcal{P}_f . This ends the proof of the claim.

Now let \mathfrak{C} be the union of \widetilde{C} for all the components C of \mathfrak{C} . Then it is a multi-annulus homotopic to \mathfrak{C} rel \mathcal{P}_f . Set \mathcal{A} to be the union of $h(\widetilde{C})$ for all the components C of \mathfrak{C} . It is also a multi-annulus homotopic to \mathfrak{C} rel \mathcal{P}_f .

For each component C^1 of \mathfrak{C}^1 , there are two distinct components E_+^1, E_-^1 of $F^{-1}(\mathcal{E})$ such that $C^1 = A(E_+^1, E_-^1)$. As above, define $\widetilde{C}^1 := A(\widehat{E}_+^1, \widehat{E}_-^1)$. It is an annulus essentially contained in C^1 . Moreover, the following statements hold:

- $h^{-1}(h(\widetilde{C}^1)) = \widetilde{C}^1$.
- $\widetilde{C}^1 \cap \widehat{E} = \emptyset$ for any $E \subset \widehat{\mathbb{C}}$ with $E \cap \widetilde{C}^1 = \emptyset$.
- $h(\widetilde{C}^1)$ is an annulus homotopic to C^1 rel \mathcal{P}_f .

Set \mathfrak{C}^1 to be the union of \widetilde{C}^1 for all the components C^1 of \mathfrak{C}^1 . Then it is a multi-annulus homotopic to \mathfrak{C}^1 rel \mathcal{P}_f . Set \mathcal{A}^1 to be the union of $h(\widetilde{C}^1)$ for all the components C^1 of \mathfrak{C}^1 . Then it is also a multi-annulus homotopic to \mathfrak{C}^1 rel \mathcal{P}_f .

Note that each component of $\widetilde{\mathfrak{C}}$ is a component of $\widehat{\mathbb{C}} \setminus \widehat{\mathcal{E}}$ and each component of $\widetilde{\mathfrak{C}}^1$ is a component of $\widehat{\mathbb{C}} \setminus F^{-1}(\widehat{\mathcal{E}}) = \widehat{\mathbb{C}} \setminus \widehat{F^{-1}(\mathcal{E})}$. So $F : \mathfrak{C}^1 \rightarrow \mathfrak{C}$ is proper. Since $\widetilde{\mathfrak{C}} = h^{-1}(\mathcal{A})$ and $\widetilde{\mathfrak{C}}^1 = h^{-1}(\mathcal{A}^1)$, the map $f : \mathcal{A}^1 \rightarrow \mathcal{A}$ is also proper.

For any component E of \mathcal{E} , there is a unique component E^1 of $F^{-1}(\mathcal{E})$ such that $\partial E \subset \partial E^1$. Moreover, $E^1 \subset E$ and $E \setminus E^1$ is a disjoint union of Jordan domains in E . We claim that $\widehat{E} \setminus E = \widehat{E^1} \setminus E$.

Since $E \supset E^1$, we have $\widehat{E} \supset \widehat{E^1}$. On the other hand, any component D of $\widehat{\mathbb{C}} \setminus E$ is a Jordan domain. Assume $z \in \widehat{E} \cap D$, then $h^{-1}(h(z))$ is a full continuum intersecting ∂E by Theorem A.1 (3). Thus $h^{-1}(h(z))$ intersects ∂E^1 . Therefore $z \in \widehat{E^1}$ and hence $\widehat{E} \setminus E \subset \widehat{E^1} \setminus E$. The claim is proved.

By the claim, each component of $\partial \widetilde{\mathcal{C}}$ is a component of $\partial \widetilde{\mathcal{C}^1}$ and hence each component of ∂A for any component A of \mathcal{A} is a component of ∂A^1 for some component A^1 of \mathcal{A}^1 in A . So $f : \mathcal{A}^1 \rightarrow \mathcal{A}$ is an exact annular system satisfying the conditions of the theorem.

Now we want to show the uniqueness of \mathcal{A} . Suppose that $f : \mathcal{A}_1^1 \rightarrow \mathcal{A}_1$ is another exact annular system satisfying the conditions. Pick an essential Jordan curve γ , γ_1 in each component of \mathcal{A} and \mathcal{A}_1 , respectively. Denote by Γ and Γ_1 the union of them. Then there exist a neighborhood U of \mathcal{P}_f and a homeomorphism θ_0 of $\widehat{\mathbb{C}}$ such that $\theta_0(\Gamma) = \Gamma_1$ and θ_0 is homotopic to the identity rel $\mathcal{P}_f \cup U$. By Theorem A.1, there exist a neighborhood V of \mathcal{P}_f and a sequence $\{\theta_n\}$ ($n \geq 1$) of homeomorphisms of $\widehat{\mathbb{C}}$ each homotopic to the identity rel $\mathcal{P}_f \cup U$, such that $f \circ \theta_n = \theta_{n-1} \circ F$. The sequence $\{\theta_n\}$ converges uniformly to a continuous map h of $\widehat{\mathbb{C}}$.

It is easy to see that h is the identity in the Fatou set of f . On the other hand, h is also the identity on the Julia set \mathcal{J}_f since the closure of $\cup_{n \geq 0} f^{-n}(\mathcal{P}_f)$ contains \mathcal{J}_f and θ_n is the identity on $f^{-n}(\mathcal{P}_f)$. So $\{\theta_n\}$ converges uniformly to the identity.

For each component A of \mathcal{A} , set $A(n, \Gamma)$ to be the closed annulus bounded by two curves in $A \cap (f|_{\mathcal{A}^1})^{-n}(\Gamma)$ such that $A \cap (f|_{\mathcal{A}^1})^{-n}(\Gamma) \subset A(n, \Gamma)$. Then $\theta_n(A(n, \Gamma)) \subset \mathcal{A}_1$ since $\theta_n(f^{-n}(\Gamma)) = f^{-n}(\Gamma_1)$. By Corollary 3.3, for any compact set $G \subset A$, $G \subset A(n, \Gamma)$ as n is large enough. So $A \subset \mathcal{A}_1$ since $\{\theta_n\}$ converges uniformly to the identity. By symmetry, we have $\mathcal{A} = \mathcal{A}_1$. \square

As a consequence of Theorem 3.7, we have

Proposition 4.5. *Let f be a postcritically finite rational map. Suppose that \mathcal{A} and \mathcal{A}^1 are multi-annuli in $\widehat{\mathbb{C}} \setminus \mathcal{P}_f$ and $g = f|_{\mathcal{A}^1} : \mathcal{A}^1 \rightarrow \mathcal{A}$ is an exact annular system. Then $\mathcal{J}_g \subset \mathcal{J}_f$ and every components of \mathcal{J}_g are Jordan curves.*

Proof. At first we prove $\mathcal{J}_g \subset \mathcal{J}_f$. Assume by contradiction that there is a point $z \in \mathcal{J}_g \setminus \mathcal{J}_f$. Then $\{f^n(z)\}_{n \geq 0}$ converges to a super-attracting cycle of f as $n \rightarrow \infty$. But $f^n(z) \in g^n(\mathcal{J}_g) \subset \mathcal{J}_g$. Thus $\partial \mathcal{A}$ contains a super-attracting point since any super-attracting point of f is contained in \mathcal{P}_f and hence can not be contained in \mathcal{A} . By the exactness of the annular system $g : \mathcal{A}^1 \rightarrow \mathcal{A}$, we know $f(\partial \mathcal{A}) \subset \partial \mathcal{A}$. Therefore $\partial \mathcal{A}$ contains a super-attracting cycle.

Let $z_0 \in \partial \mathcal{A}$ be a super-attracting point of f with period $p \geq 1$. Then there is a disk U containing z_0 such that $f^p(U) \subset \subset U$. Let A^n be a component of $g^{-n}(\mathcal{A})$ such that $z_0 \in \partial A^n$ and $A^{n+1} \subset A^n$. Then there are integers $m \geq 0$ and $k \geq 1$ such that $g^{kp}(g^{mp}(A^{kp+mp})) = g^{mp}(A^{mp})$ since \mathcal{A} has only finitely many components.

Since $f^{np}(U)$ converges uniformly to the point z_0 , there is an integer $n_0 \geq 0$ such that as $n \geq n_0$, $f^{np}(U)$ is disjoint from the component of $\partial g^{mp}(A^{mp})$ which does not contain z_0 . On the other hand, since $\bigcap_{n \geq 0} A^n = \emptyset$ by Proposition 3.2, there is an integer $l > n_0/k$ such that both components of $\partial g^{mp}(A^{lkp+mp})$ intersect

U . Thus both components of $\partial g^{mp}(A^{mp})$ intersect $f^{lkp}(U)$. Since $lk > n_0$, this is a contradiction.

There is a singular conformal metric ρ on $\widehat{\mathbb{C}}$ where the singularities may occur at \mathcal{P}_f such that f is expanding on $(\widehat{\mathbb{C}}, \rho)$ (for example, the hyperbolic metric on the orbifold of f , refer to [8] or [26]). Applying Theorem 3.7, we see that every components of \mathcal{J}_g are Jordan curves. \square

5 Renormalizations

Definition 5. Let $U \subset\subset V$ be two connected and finitely-connected domains in $\widehat{\mathbb{C}}$. We say that a map $g : U \rightarrow V$ is a **rational-like map** if

- (1) g is holomorphic, proper and $\deg g \geq 2$,
- (2) the orbit of every critical point of g (if any) stays in U , and
- (3) each component of $\widehat{\mathbb{C}} \setminus U$ contains at most one component of $\widehat{\mathbb{C}} \setminus V$ (see Figure 1).

The **filled-in Julia set** of g is defined by $\mathcal{K}_g = \bigcap_{n \geq 0} g^{-n}(V)$.

We say that a rational-like map $g : U \rightarrow V$ is a **renormalization** of a rational map f if $g = f^p|_U$ for some $p \geq 1$ and $\deg g < \deg f^p$.

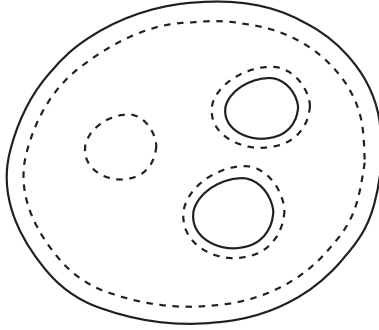


Figure 1 The domain bounded by dotted lines (resp. solid lines) is U (resp. V).

Remark. 1. A rational-like map here is actually a repelling system of constant complexity in [6].

2. In Figure 2, the picture at the top is the Julia set of the quadratic polynomial $z \mapsto z^2 - 1$ and the one at the bottom is the Julia set of the rational map $z \mapsto \frac{1}{z} \circ (z^2 - 1) \circ \frac{1}{z} + 10^{-11}z^{-3}$, in $\log(z)$ -coordinates. This figure shows that in a rational map one has a polynomial-like renormalization (see [21] for details). One can also refer to Sebastien's thesis ([12]) for a rational-like but non polynomial-like renormalization.

Obviously, \mathcal{K}_g is a compact set. Similar to Douady-Hubbard's polynomial-like map theory ([10]), there is a Straightening Theorem for rational-like maps with essentially the same proof.

Theorem 5.1. Let $g : U \rightarrow V$ be a rational-like map, then there is a rational map f and a quasiconformal map ϕ of $\widehat{\mathbb{C}}$ such that:

- (a) $f \circ \phi = \phi \circ g$ on a neighborhood of \mathcal{K}_g ,
- (b) ϕ has a complex dilatation μ_ϕ satisfying $\mu_\phi(z) = 0$ for a.e. $z \in \mathcal{K}_g$,
- (c) \mathcal{K}_g is connected and $\mathcal{J}_f = \partial\phi(\mathcal{K}_g)$, and
- (d) each component of $\widehat{\mathbb{C}} \setminus \phi(\mathcal{K}_g)$ contains at most one point of \mathcal{P}_f .

Moreover, the rational map f is unique up to holomorphic conjugation.

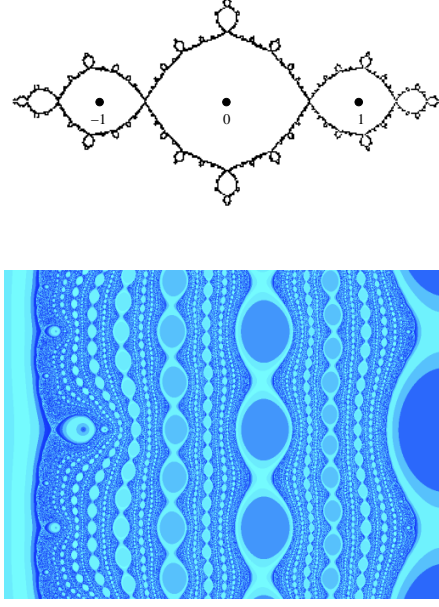


Figure 2

Proof. Pick a domain $V_1 \subset\subset V$ such that every component of $\widehat{\mathbb{C}} \setminus V_1$ contains exactly one component of $\widehat{\mathbb{C}} \setminus V$, $U \subset\subset V_1$ and every component of ∂V_1 is a quasicircle. Then $U_1 := g^{-1}(V_1) \subset\subset V_1$, every component of $\widehat{\mathbb{C}} \setminus U_1$ contains at most one component of $\widehat{\mathbb{C}} \setminus V_1$ and each component of ∂U_1 is a quasicircle.

Let E_1, \dots, E_m be the components of $\mathcal{E} := \widehat{\mathbb{C}} \setminus \overline{V_1}$. Let B_1, \dots, B_n be the components of $\mathcal{B} := \widehat{\mathbb{C}} \setminus \overline{U_1}$ such that $B_i \supset E_i$ for $1 \leq i \leq m$. Then $\mathcal{E} \subset\subset \mathcal{B}$. Define a map τ on the index set by $\tau(i) = j$ if $g(\partial B_i) = \partial E_j$.

Let $D_i \subset \mathbb{C}$ ($i = 1, \dots, n$) be round disks centered at a_i with unit radius such that their closures are pairwise disjoint. Denote their union by \mathcal{D} . Define a map Q on \mathcal{D} by

$$Q(z) = r(z - a_i)^{d_i} + a_{\tau(i)}, \quad z \in D_i,$$

where $0 < r < 1$ is a constant and $d_i = \deg(g|_{\partial E_i})$. Then $D_{\tau(i)}(r) := Q(D_i) \subset\subset D_{\tau(i)}$. Denote $\mathcal{D}(r) = Q(\mathcal{D})$.

Let $\psi : \mathcal{E} \rightarrow \mathcal{D}(r)$ be a conformal map such that $\psi(E_i) = D_i(r)$. It can be extended to a quasiconformal map on a neighborhood of $\overline{\mathcal{E}}$ since every components of \mathcal{E} are quasidisks whose closures are disjoint. Since $Q : \partial \mathcal{D} \rightarrow \partial \mathcal{D}(r)$ and $g : \partial \mathcal{B} \rightarrow \partial \mathcal{E}$ are coverings with same degrees on corresponding components, there is a homeomorphism $\psi_1 : \partial \mathcal{B} \rightarrow \partial \mathcal{D}$ such that $\psi \circ g = Q \circ \psi_1$.

Since each component of $\partial \mathcal{B}$ is a quasicircle, the conformal map $\psi : \mathcal{E} \rightarrow \mathcal{D}(r)$ can be extended to a homeomorphism $\psi : \overline{\mathcal{B}} \rightarrow \overline{\mathcal{D}}$ such that $\psi|_{\partial \mathcal{B}} = \psi_1$ and ψ is quasiconformal on \mathcal{B} . Define a map

$$G = \begin{cases} g & \text{on } U_1, \\ \psi^{-1} \circ Q \circ \psi & \text{on } \overline{\mathcal{B}}. \end{cases}$$

Then G is a quasiregular branched covering of $\widehat{\mathbb{C}}$. Set $\mathcal{O} := \psi^{-1}(\{a_1, \dots, a_n\})$. Then

$$G(\mathcal{O}) \subset \mathcal{O} \quad \text{and} \quad \mathcal{P}_G \setminus \mathcal{K}_g \subset \mathcal{O}$$

since no critical point of g escapes. Moreover, for each point $z \in \widehat{\mathbb{C}} \setminus \mathcal{K}_g$, its forward orbit $\{G^n(z)\}$ converges to the invariant set \mathcal{O} .

By the measurable Riemann mapping theorem, there is a quasiconformal map Φ of $\widehat{\mathbb{C}}$ such that its complex dilatation satisfies $\mu_\Phi = 0$ on U_1 and $\mu_\Phi = \mu_\psi$ on \mathcal{B} . Set $F := \Phi \circ G \circ \Phi^{-1}$. Then F is holomorphic in $\Phi(g^{-1}(U_1) \cup \mathcal{E})$. It is easy to check that every orbit of F passes through the remaining subset at most three times. Applying Shishikura's Surgery Principle (see Lemma 15 in [1]), there is quasiconformal map $\Phi_1 : \widehat{\mathbb{C}} \rightarrow \widehat{\mathbb{C}}$ such that $f = \Phi_1 \circ F \circ \Phi_1^{-1}$ is a rational map. Moreover, $\mu_{\Phi_1}(z) = 0$ for *a.e.* $z \in \Phi(\mathcal{K}_g)$. Set $\phi = \Phi_1 \circ \Phi$. Then $f \circ \phi = \phi \circ g$ on U_1 and $\mu_\phi(z) = 0$ for *a.e.* $z \in \mathcal{K}_g$.

For a compact set $E \subset \widehat{\mathbb{C}} \setminus \phi(\mathcal{K}_g)$, its forward orbit $\{f^n(E)\}$ converges to the invariant set $\phi(\mathcal{O}) \subset \mathcal{F}_f$. Moreover, $\mathcal{P}_f \setminus \phi(\mathcal{K}_g) \subset \phi(\mathcal{O})$. So $\widehat{\mathbb{C}} \setminus \phi(\mathcal{K}_g) \subset \mathcal{F}_f$ and each component of $\widehat{\mathbb{C}} \setminus \phi(\mathcal{K}_g)$ is simply-connected. Thus $\phi(\mathcal{K}_g)$ is connected. Since $\phi(\mathcal{K}_g)$ is completely invariant under f , its boundary is the Julia set of f .

If there is another rational map f_1 satisfying the conditions of the theorem, then there is a quasiconformal map θ of $\widehat{\mathbb{C}}$ such that $f_1 \circ \theta = \theta \circ f$ in a neighborhood of $\phi(\mathcal{K}_g)$ and $\mu_\theta(z) = 0$ for *a.e.* $z \in \phi(\mathcal{K}_g)$.

Let W be a periodic Fatou domain of f in $\widehat{\mathbb{C}} \setminus \phi(\mathcal{K}_g)$ with period $p \geq 1$. Then W is simply-connected and contains exactly one point $z_0 \in \mathcal{P}_f$, which is the super-attracting periodic point. Therefore there is a conformal map η from W onto the unit disc \mathbb{D} such that $\eta(z_0) = 0$ and $\eta \circ f^p \circ \eta^{-1}(z) = z^d$ with $d = \deg_{z_0} f^p > 1$. On the other hand, let $z_1 \in \theta(W)$ be the super-attracting periodic point of f_1 , then there is a conformal map $\eta_1 : \theta(W) \rightarrow \mathbb{D}$ such that $\eta_1(z_1) = 0$ and $\eta_1 \circ f_1^p \circ \eta_1^{-1}(z) = z^d$. Therefore

$$\eta_1 \circ \theta \circ f^p \circ \theta^{-1} \circ \eta^{-1}(z) = z^d$$

in a neighborhood of $\partial\mathbb{D}$ in \mathbb{D} . This shows that $T = \eta_1 \circ \theta \circ \eta^{-1}$ is a rotation on $\partial\mathbb{D}$ (see the commutative diagram below).

$$\begin{array}{ccccccc} \mathbb{D} & \xleftarrow{\eta} & W & \xrightarrow{\theta} & \theta(W) & \xrightarrow{\eta_1} & \mathbb{D} \\ z \mapsto z^d \downarrow & & f^p \downarrow & & \downarrow f_1^p & & \downarrow z \mapsto z^d \\ \mathbb{D} & \xleftarrow{\eta} & W & \xrightarrow{\theta} & \theta(W) & \xrightarrow{\eta_1} & \mathbb{D} \end{array}$$

Let $\theta_W = \eta_1^{-1} \circ T \circ \eta$. Then $\theta_W : W \rightarrow \theta(W)$ is holomorphic, $\theta_W = \theta$ on the boundary ∂W and $f_1 \circ \theta_W = \theta_W \circ f$.

Define $\Theta_0 : \widehat{\mathbb{C}} \rightarrow \widehat{\mathbb{C}}$ by $\Theta_0 = \theta_W$ on all the super-attracting Fatou domains of f in $\widehat{\mathbb{C}} \setminus \phi(\mathcal{K}_g)$, and $\Theta_0 = \theta$ otherwise. Then Θ_0 is a quasiconformal map and $\Theta_0 \circ f = f_1 \circ \Theta_0$ on the union of $\phi(\mathcal{K}_g)$ and all the super-attracting Fatou domains of f in $\widehat{\mathbb{C}} \setminus \phi(\mathcal{K}_g)$. Pullback Θ_0 , we have a sequence of quasiconformal maps $\Theta_n : \widehat{\mathbb{C}} \rightarrow \widehat{\mathbb{C}}$ such that $\Theta_0 \circ f^n = f_1^n \circ \Theta_n$, in particular, the following diagram commutes.

$$\begin{array}{ccc} \vdots & & \vdots \\ f \downarrow & & \downarrow f_1 \\ \widehat{\mathbb{C}} & \xrightarrow{\Theta_n} & \widehat{\mathbb{C}} \\ f \downarrow & & \downarrow f_1 \\ \vdots & & \vdots \\ f \downarrow & & \downarrow f_1 \\ \widehat{\mathbb{C}} & \xrightarrow{\Theta_1} & \widehat{\mathbb{C}} \\ f \downarrow & & \downarrow f_1 \\ \widehat{\mathbb{C}} & \xrightarrow{\Theta_0} & \widehat{\mathbb{C}} \end{array}$$

It is easy to check that Θ_n uniformly converges to a holomorphic conjugacy from f to f_1 . \square

The next theorem is a more precise version of Theorem 1.2.

Theorem 5.2. *Let f be a postcritically finite rational map with a stable Cantor multicurve Γ . Then there is a domain $W \subset \widehat{\mathbb{C}}$ such that:*

- (1) *Each component of ∂W is a non-peripheral Jordan curve in $\widehat{\mathbb{C}} \setminus \mathcal{P}_f$ homotopic rel \mathcal{P}_f to a curve in Γ , and no two curves on ∂W are homotopic rel \mathcal{P}_f .*
- (2) *There are integers $k, p \geq 1$, a component V of $f^{-kp}(W)$ and a component U of $f^{-p}(V)$ such that $f^p : U \rightarrow V$ is a renormalization of f .*

Proof. By Theorem 4.1, there is a multi-annulus $\mathcal{A} \subset \widehat{\mathbb{C}}$ such that \mathcal{A} is homotopic to Γ rel \mathcal{P}_f and $f : \mathcal{A}^1 \rightarrow \mathcal{A}$ is an exact annular system, where \mathcal{A}^1 is the union of components of $f^{-1}(\mathcal{A})$ homotopic to curves in Γ . Set $\mathcal{A}^n = (f|_{\mathcal{A}^1})^{-n}(\mathcal{A})$. Since Γ is stable, for every $n \geq 1$ and every component A^n of $f^{-n}(\mathcal{A}) \setminus \mathcal{A}^n$, one component of $\widehat{\mathbb{C}} \setminus \mathcal{A}^n$ is disjoint from $\partial \mathcal{A}$.

Set $\mathcal{B} := \widehat{\mathbb{C}} \setminus \mathcal{A} = B_0^0 \cup \dots \cup B_m^0$ to be the disjoint union of the continua B_i^0 . Then each B_i^0 is also a component of $\widehat{\mathbb{C}} \setminus \mathcal{A}^n$ for $n \geq 1$ since $\partial \mathcal{A} \subset \partial \mathcal{A}^n$. Noticing that

$$f^{-n}(\mathcal{B}) = (\widehat{\mathbb{C}} \setminus \mathcal{A}^n) \setminus (f^{-n}(\mathcal{A}) \setminus \mathcal{A}^n),$$

each annulus of $f^{-n}(\mathcal{A}) \setminus \mathcal{A}^n$ is either disjoint from B_i^0 or contained in B_i^0 . In the latter case it splits B_i^0 into two continua and one of them is disjoint from $\partial \mathcal{A}$. Therefore B_i^0 contains exactly one component of $f^{-n}(\mathcal{B})$, denoted by B_i^n , such that each component of $B_i^0 \setminus B_i^n$ is a simply-connected domain disjoint from $\partial \mathcal{A}$.

Define a map τ_f from the index set $\{0, \dots, m\}$ to itself such that $f(B_i^1) = B_{\tau_f(i)}^0$. Then each index is eventually periodic under the map τ_f . Therefore there is an index such that it is periodic under τ_f . Without loss of generality, we may assume that the index 0 is periodic with period $p \geq 1$, i.e. $B_0^{(k+1)p}$ is a component of $f^{-p}(B_0^{kp})$ for all $k \geq 0$.

Let W' be the union of B_0^0 together with the annuli of \mathcal{A} whose boundary intersects B_0^0 . It is a finitely-connected domain. Let W'_1 be the component of $f^{-p}(W')$ containing B_0^p . Then $W'_1 \subset W'$ and each component of $\widehat{\mathbb{C}} \setminus W'_1$ contains at most one component of $\widehat{\mathbb{C}} \setminus W'$.

Now let A_1, \dots, A_n be the components of $W' \setminus B_0^0$. They are annuli of \mathcal{A} . Then each A_i contains exactly one component of $W'_1 \setminus B_0^p$, denoted by A_i^p , which is a component of \mathcal{A}^p and shares a common boundary component with A_i . As above, each A_i is pre-periodic in the sense that there are integers $k \geq 0$ and $q \geq 1$ such that $f^{qp}(f^{kp}(A_i^{(k+q)p})) = f^{kp}(A_i^{kp})$.

Assume that $\{A_1, \dots, A_q\}$ is a periodic cycle with period $q \geq 1$, i.e. $f^p(A_i^p) = A_{i+1}$ for $i = 1, \dots, q-1$ and $f^p(A_q^p) = A_1$. Then at least one of them, say A_1 satisfying $A_1^p \neq A_1$. Otherwise $A_1^{qp} = A_1$ and hence $f^{qp}(A_1) = A_1$, contradicting the fact that $f : \mathcal{A}^1 \rightarrow \mathcal{A}$ is an annular system.

We may now assume $A_1^p \neq A_1$. Then there is a Jordan curve γ_1 essentially contained in A_1 disjoint from A_1^p . Let γ'_q be the component of $f^{-p}(\gamma_1)$ in A_q^p , then we can find a Jordan curve γ_q essentially contained in A_q and disjoint from γ'_q such that γ'_q separates γ_q from B_0^0 . Inductively, for $2 \leq i < q$, we can find a Jordan curve γ_i essentially contained in A_i such that the component of $f^{-p}(\gamma_{i+1})$ in A_i separates γ_i from B_0^0 . Since γ_1 is disjoint from A_1^p , the component of $f^{-p}(\gamma_2)$ in A_1 separates γ_1 from B_0^0 as well.

Do this process for each cycle, we have a Jordan curve γ_i essentially contained in each periodic annulus A_i such that if $f^p(A_i^p) = A_j$, and the component of $f^{-p}(\gamma_j)$ in A_i separates γ_i from B_0^0 . If A_i is not periodic but $A_j = f^p(A_i^p)$ is periodic, then there is always a Jordan curve γ_i essentially contained in A_i such that the component of $f^{-p}(\gamma_j)$ in A_i separates γ_i from B_0^0 .

In summary, we can find a curve $\gamma_i \subset A_i$ for each component A_i of $W' \setminus B_0^0$ such that if $f^p(A_i^p) = A_j$, then the component of $f^{-p}(\gamma_j)$ in A_i^p separates γ_i from B_0^0 . Let $W \subset W'$ be the domain bounded by the curves γ_i defined above. Then $W_1 \subset\subset W$ where W_1 is the component of $f^{-p}(W)$ containing B_0^p . Clearly, each component of $\widehat{\mathbb{C}} \setminus W_1$ contains at most one component of $\widehat{\mathbb{C}} \setminus W$.

Let W_n be the component of $f^{-np}(W)$ containing B_0^{np} for $n \geq 2$. Then $W_n \subset\subset W_{n-1}$. Since each component of $W \setminus \overline{W_1}$ is either an annulus homotopic to a curve in the stable multicurve Γ , or a disk which contains at most one point of \mathcal{P}_f , their pre-images are either disks or annuli whose essential curves are either non-essential or peripheral or homotopic to a curve in Γ . From this fact one can easily check that each component of $\widehat{\mathbb{C}} \setminus W_n$ contains at most one component of $\widehat{\mathbb{C}} \setminus W_{n-1}$.

Since \mathcal{P}_f is finite, there is an integer $N \geq 1$ such that $W_n \cap \mathcal{P}_f = W_N \cap \mathcal{P}_f$ for all $n \geq N$. Set $U = W_{N+1}$, $V = W_N$ and $g := f^p : U \rightarrow V$. By Proposition 3.1, there is an integer $n \geq 1$ such that $\deg f^n|_U \geq 2$ for all the components U of \mathcal{A}^n . So we have $\deg g \geq 2$. Therefore $g := f^p : U \rightarrow V$ is a rational-like map since no critical point of g escapes.

We show now $\deg g < \deg f^p$. Otherwise $\mathcal{J}_f \subset \mathcal{K}_g \subset \mathcal{B}$. But we know that the Julia set of the annular system $f : \mathcal{A}^1 \rightarrow \mathcal{A}$ is contained in \mathcal{J}_f . This is not possible. So $\deg g < \deg f^p$ and consequently g is a renormalization of f . \square

6 Wandering continua

Definition 6. Let f be a rational map. By a **wandering continuum** we mean a non-degenerate continuum $K \subset \mathcal{J}_f$ (i.e. K is a connected compact set consisting more than one point) such that $f^n(K) \cap f^m(K) = \emptyset$ for any $n > m \geq 0$.

The existence or not of wandering continua for polynomials has been studied by many authors (refer to [27, 16, 3, 14, 15]). It is proved that for a polynomial without irrational indifferent periodic cycles, there is no wandering continuum on the Julia set if and only if the Julia set is locally connected ([14, 3]).

We say that a continuum $E \subset \widehat{\mathbb{C}} \setminus \mathcal{P}_f$ is **non-peripheral** if there is an annulus A disjoint from \mathcal{P}_f such that E is essentially contained in A and each complementary component of A contains at least two points of \mathcal{P}_f . In this case, we say that E is homotopic rel \mathcal{P}_f to a curve γ on $\widehat{\mathbb{C}} \setminus \mathcal{P}_f$ if A is homotopic to γ rel \mathcal{P}_f .

Proposition 6.1. Let f be a postcritically finite rational map. Suppose that $K \subset \mathcal{J}_f$ is a wandering continuum. Then either

- (1) $f^n(K)$ is full (i.e. $\widehat{\mathbb{C}} \setminus K$ is connected) for all $n \geq 0$; or
- (2) there exists an integer $N \geq 0$ such that for $n \geq N$, $f^n(K)$ is non-peripheral.

The wandering continuum K is said to be a **full wandering continuum** in the first case; or a **separating wandering continuum** in the second case.

Proof. Set $K_n := f^n(K)$ for $n \geq 0$. Since $\#\mathcal{P}_f < \infty$ and the K_n 's are pairwise disjoint, we have $K_n \cap \mathcal{P}_f = \emptyset$ for all $n \geq 0$.

Suppose that K is not a full wandering continuum, i.e., there is an integer $n_0 \geq 1$ such that K_{n_0} is not full, then K_n is not full for all $n \geq n_0$. Otherwise, assume that there is an integer $m > n_0$ such that K_m is full, then there is a disk D containing K_m such that $D \cap \mathcal{P}_f = \emptyset$. Let D_n be the component of $f^{n-m}(D)$ containing K_n for $0 \leq n \leq m$. Then $f^n : D_n \rightarrow D$ is a homeomorphism. So K_n is also full. This contradicts that K_{n_0} is not full.

Let $s(K_n) \geq 1$ be the number of components of $\widehat{\mathbb{C}} \setminus K_n$ containing points of \mathcal{P}_f . Since K_n are pairwise disjoint, there are at most $(\#\mathcal{P}_f - 2)$ continua K_n with $s(K_n) \geq 3$. Thus there is an integer $n_1 \geq n_0$ such that $s(K_n) \leq 2$ for all $n \geq n_1$.

If $s(K_n) \equiv 1$ for all $n \geq n_1$, let \widehat{K}_n be the union of K_n together with the components of $\widehat{\mathbb{C}} \setminus K_n$ disjoint from \mathcal{P}_f , then $f : \widehat{K}_n \rightarrow \widehat{K}_{n+1}$ is a homeomorphism for $n \geq n_1$. Since K_{n_1} is not full, $\widehat{K}_{n_1} \setminus K_{n_1}$ is non-empty. Let U be a component of $\widehat{K}_{n_1} \setminus K_{n_1}$. Then $U \cap \mathcal{P}_f = \emptyset$. If $U \cap \mathcal{J}_f \neq \emptyset$, then $f^m(U) \supset \mathcal{J}_f$ for some $m \geq 1$. But $f^m(U)$ is a component of $\widehat{\mathbb{C}} \setminus K_{n_1+m}$. This is a contradiction. So $U \cap \mathcal{J}_f = \emptyset$. Noticing that $\partial U \subset K_{n_1} \subset \mathcal{J}_f$, the simply-connected domain U is a Fatou domain and ∂U is wandering. This contradicts the no wandering Fatou domain theorem of Sullivan (refer to [19]). Therefore there is an integer $n_2 \geq n_1$ such that $s(K_{n_2}) = 2$.

We claim that $s(K_n) \equiv 2$ for all $n \geq n_2$. Otherwise, assume that there is an integer $m > n_2$ such that $s(K_m) = 1$, then there is a disk D containing K_m such that $D \cap \mathcal{P}_f = \emptyset$. Let D_n be the component of $f^{n-m}(D)$ containing K_n for $n_2 \leq n \leq m$. Then D_n is disjoint from \mathcal{P}_f . So $s(K_n) = 1$ for $n_2 \leq n \leq m$. This contradicts $s(K_{n_2}) = 2$.

We may assume $\#\mathcal{P}_f \geq 3$ (otherwise f is conjugate to the maps $z \rightarrow z^{\pm d}$ and hence has no wandering continuum), then f has at most one exceptional point. If there is an integer $m \geq n_2$ such that $\widehat{\mathbb{C}} \setminus K_m$ has a component containing exactly one \mathcal{P}_f point, then there is a disk $D \supset K_m$ such that D contains exactly one \mathcal{P}_f point. Let D_n be the component of $f^{n-m}(D)$ containing K_n for $n_2 \leq n \leq m$. Then D_n is simply-connected and contains at most one point of \mathcal{P}_f . Thus $\widehat{\mathbb{C}} \setminus K_n$ has a component containing exactly one \mathcal{P}_f point for $n_2 \leq n \leq m$. Therefore either there exists an integer $N \geq n_2$ such that for $n \geq N$, $f^n(K)$ is non-peripheral, or $\widehat{\mathbb{C}} \setminus f^n(K)$ has a component containing exactly one \mathcal{P}_f point for all $n \geq n_2$.

In the latter case, denote by U the component of $\widehat{\mathbb{C}} \setminus K_{n_2}$ containing exactly one \mathcal{P}_f point. If $U \cap \mathcal{J}_f \neq \emptyset$, then there is an integer $k > 0$ such that $\widehat{\mathbb{C}} \setminus f^k(U)$ contains at most one point (an exceptional point). On the other hand, there is a disk $D \supset K_{n_2+k}$ such that D contains exactly one \mathcal{P}_f point. Let D_{n_2} be the component of $f^{-k}(D)$ containing K_{n_2} . Then D_{n_2} is simply-connected and contains at most one point of \mathcal{P}_f . Thus $U \subset D_{n_2}$. Therefore $f^k(U) \subset D$ and hence $\widehat{\mathbb{C}} \setminus D \subset \widehat{\mathbb{C}} \setminus f^k(U)$ contains at most one point. This contradicts $\#\mathcal{P}_f \geq 3$. So U is disjoint from \mathcal{J}_f and hence is a simply-connected Fatou domain. This again contradicts Sullivan's no wandering Fatou domain theorem. \square

Lemma 6.2. *Suppose that $K \subset \mathcal{J}_f$ is a separating wandering continuum. Then there is a multicurve Γ_K such that:*

- (1) *for each curve γ in Γ_K , there are infinitely many continua $f^n(K)$ which are non-peripheral and homotopic to γ rel \mathcal{P}_f , and*
- (2) *there is an integer $N_1 \geq 0$ such that for $n \geq N_1$, $f^n(K)$ is non-peripheral and homotopic rel \mathcal{P}_f to a curve in Γ_K .*

The multicurve Γ_K is uniquely determined by K up to homotopy. We call it the multicurve generated by K .

Proof. By Lemma 6.1, there is an integer $N \geq 0$ such that $f^n(K)$ is non-peripheral for $n \geq N$. Since the $f^n(K)$'s are pairwise disjoint, for any integer $m \geq N$, we may choose a non-peripheral Jordan curve β_n on $\widehat{\mathbb{C}} \setminus \mathcal{P}_f$ for $N \leq n \leq m$ such that they are pairwise disjoint and $f^n(K)$ is homotopic to β_n rel \mathcal{P}_f . Let Γ_m be the collection of these curves β_n . Let $\tilde{\Gamma}_m \subset \Gamma_m$ be a multicurve such that each curve in Γ_m is homotopic to a curve in $\tilde{\Gamma}_m$. Then each curve in $\tilde{\Gamma}_m$ is homotopic to a curve in $\tilde{\Gamma}_{m+1}$. This implies that $\#\tilde{\Gamma}_m$ is increasing and hence there is an integer $m_0 \geq N$ such that $\#\tilde{\Gamma}_m$ is a constant for $m \geq m_0$ since any multicurve contains at most $\#\mathcal{P}_f - 3$ curves. Therefore each curve in $\tilde{\Gamma}_{m+1}$ is homotopic to a curve in $\tilde{\Gamma}_m$ for $m \geq m_0$. This shows that the multicurves $\tilde{\Gamma}_m$ are homotopic to each other for all $m \geq m_0$.

Let $\Gamma_K \subset \tilde{\Gamma}_{m_0}$ be the sub-collection consisting of curves $\gamma \in \tilde{\Gamma}_{m_0}$ such that there are infinitely many $f^n(K)$ homotopic to γ rel \mathcal{P}_f . It is easy to see that it is non-empty and hence is a multicurve. Obviously, Γ_K is uniquely determined by K and there is an integer $N_1 \geq 0$ such that for $n \geq N_1$, $f^n(K)$ is non-peripheral and homotopic rel \mathcal{P}_f to a curve in Γ_K . \square

Lemma 6.3. Γ_K is an irreducible Cantor multicurve.

Proof. By Lemma 6.2, there exists an integer $N_1 \geq 0$ such that $f^n(K)$ for $n \geq N_1$ is non-peripheral and homotopic to a curve in Γ_K rel \mathcal{P}_f . For any pair $(\gamma, \alpha) \in \Gamma_K \times \Gamma_K$, there are integers $N_1 \leq k_1 < k_2$ such that $f^{k_1}(K)$ is homotopic to γ and $f^{k_2}(K)$ is homotopic to α rel \mathcal{P}_f . Thus $f^{k_1-k_2}(\alpha)$ has a component δ homotopic to γ rel \mathcal{P}_f . So for $1 < i < k_2 - k_1$ the curve $f^i(\delta)$ is homotopic rel \mathcal{P}_f to $f^{k_1+i}(K)$ and hence to a curve in Γ_K rel \mathcal{P}_f . This shows that Γ_K is irreducible.

Let us now prove that Γ_K is a Cantor multicurve. We may apply Lemma 2.1 and assume by contradiction that $f^{-1}(\gamma)$ for each $\gamma \in \Gamma_K$ has exactly one component homotopic rel \mathcal{P}_f to a curve in Γ_K . Assume $N_1 = 0$ for simplicity. Denote by $\Gamma_K = \{\gamma_0, \dots, \gamma_{p-1}\}$ such that γ_0 is homotopic to K and γ_n is homotopic to a component of $f^{-1}(\gamma_{n+1})$ for $0 \leq n < p$ (set $\gamma_p = \gamma_0$). It makes sense since each $\gamma \in \Gamma_K$ has exactly one component homotopic rel \mathcal{P}_f to a curve in Γ_K . Then $f^n(K)$ is homotopic to γ_k if $n \equiv k \pmod{p}$.

For $n \geq 0$ and $k \geq 1$ denote by $A(n, n+kp)$ the unique annular component of $\widehat{\mathbb{C}} \setminus (f^n(K) \cup f^{n+kp}(K))$. Then $f^m : A(n, n+kp) \rightarrow A(n+m, n+kp+m)$ is proper. This is because that $A(n+m, n+kp+m)$ is disjoint from \mathcal{P}_f and homotopic to $f^{n+m}(K)$, so $f^{-m}(A(n+m, n+kp+m))$ has a unique component homotopic to $f^n(K)$. This unique component must be $A(n, n+kp)$. Choose (n, k) such that $A(n, n+kp)$ contains \mathcal{J}_f points. Then $\widehat{\mathbb{C}} \setminus f^m(A(n, n+kp+m))$ contains at most one point when m is large enough. This is a contradiction. \square

Theorem 6.4. Let f be a postcritically finite rational map. If Γ is a Cantor multicurve, then $\mathcal{J}_f \setminus \mathcal{P}_f$ contains a Jordan curve as a separating wandering continuum. Conversely, if K is a separating wandering continuum, then Γ_K , the multicurve generated by K , is an irreducible Cantor multicurve. Moreover, $f^n(K)$ is a Jordan curve for every large enough n .

Proof. Let Γ be a Cantor multicurve of f . Then by Theorem 4.1, there is a multi-annulus $\mathcal{A} \subset \widehat{\mathbb{C}} \setminus \mathcal{P}_f$ homotopic rel \mathcal{P}_f to Γ such that $g = f|_{\mathcal{A}^1} : \mathcal{A}^1 \rightarrow \mathcal{A}$ is an exact annular system, where \mathcal{A}^1 is the union of components of $f^{-1}(\mathcal{A})$ homotopic rel \mathcal{P}_f to curves in Γ . By Corollary 3.3, there is a component K of \mathcal{J}_g such that K is wandering. Applying Theorem 4.5, we see that $K \subset \mathcal{J}_f$ and K is a Jordan curve.

Conversely, suppose that $K \subset \mathcal{J}_f$ is a separating wandering continuum. Then Γ_K is an irreducible Cantor multicurve by Lemma 6.3. Now we only need to show that $f^n(K)$ is a Jordan curve as n is large enough.

By Lemma 6.2, there exists an integer $N_1 \geq 0$ such that $f^n(K)$ for $n \geq N_1$ is non-peripheral and homotopic to a curve in $\Gamma_K \text{ rel } \mathcal{P}_f$. We assume $N_1 = 0$ for simplicity.

Let \mathcal{E} be the collection of non-peripheral components E of $f^{-m}(f^n(K))$ such that $f^i(E)$ is homotopic to a curve in Γ_K for all $n, m \geq 0$ and $0 < i < m$. Then $f(E) \in \mathcal{E}$ for any $E \in \mathcal{E}$, and any two elements in \mathcal{E} are either disjoint or one contains another as subsets of $\widehat{\mathbb{C}}$.

For each $\gamma \in \Gamma_K$, let $\mathcal{E}(\gamma)$ be the collection of continua in \mathcal{E} homotopic to $\gamma \text{ rel } \mathcal{P}_f$. We claim that for any continuum $E \in \mathcal{E}(\gamma)$, there are two disjoint continua $E_1, E_2 \in \mathcal{E}(\gamma)$ such that $E \subset A(E_1, E_2)$, where $A(E_1, E_2)$ denotes the unique annular component of $\widehat{\mathbb{C}} \setminus (E_1 \cup E_2)$.

Consider $f^n(E)$ for $0 \leq n \leq 2 \cdot \#\Gamma_K + 1$. There is a curve $\beta \in \Gamma_K$ such that at least three of the $f^n(E)$'s are contained in $\mathcal{E}(\beta)$. Numerate them by $f^{n_i}(E)$ ($i = 1, 2, 3$) such that $f^{n_3}(E) \subset A(f^{n_1}(E), f^{n_2}(E))$. Let A be the component of $f^{-n_3}(A(f^{n_1}(E), f^{n_2}(E)))$ that contains E . Then $A = A(E_1, E_2)$ where E_i ($i = 1, 2$) is a component of $f^{-n_3}(f^{n_i}(E))$. Now the claim is proved.

Denote $A(\gamma) = \cup A(E, E')$ for all disjoint pairs $E, E' \in \mathcal{E}(\gamma)$. Then $A(\gamma)$ is an annulus in $\widehat{\mathbb{C}} \setminus \mathcal{P}_f$ homotopic to $\gamma \text{ rel } \mathcal{P}_f$, and $A(\gamma) \cap A(\beta) = \emptyset$ for distinct curves $\beta, \gamma \in \Gamma_K$.

Denote $\mathcal{A} = \cup A(\gamma)$ and \mathcal{A}^1 the union of components of $f^{-1}(\mathcal{A})$ homotopic to curves in Γ_K . Then $\mathcal{A}^1 \subset \mathcal{A}$. For any $E \in \mathcal{E}(\gamma)$ with $\gamma \in \Gamma_K$, $f(E) \in \mathcal{E}(\beta)$ for some $\beta \in \Gamma_K$. By the claim, there are two disjoint continua $E_1, E_2 \in \mathcal{E}(\beta)$ such that $f(E) \subset A(E_1, E_2)$. Therefore $E \subset \mathcal{A}^1$. So $g = f|_{\mathcal{A}^1} : \mathcal{A}^1 \rightarrow \mathcal{A}$ is an exact annular system. Obviously, $K \subset \mathcal{J}_g$. But K is non-peripheral. Therefore K is a Jordan curve by Proposition 4.5. So is $f^n(K)$ for all $n \geq 0$. \square

Example. Let f be a flexible Lattès map. For example (refer to §7 in [19]),

$$f(z) = \frac{(z^2 + 1)^2}{4z(z - 1)(z + 1)}.$$

The post-critical set \mathcal{P}_f is $\{0, 1, -1, \infty\}$.

Set $\Lambda = \mathbb{Z} \oplus i\mathbb{Z}$ and $\chi(w) = 2w$. Then there is a holomorphic branched covering $\wp : \mathbb{C}/\Lambda \rightarrow \widehat{\mathbb{C}}$ with $\deg \wp = 2$ such that the following diagram commutes.

$$\begin{array}{ccc} \mathbb{C}/\Lambda & \xrightarrow{\chi} & \mathbb{C}/\Lambda \\ \wp \downarrow & & \downarrow \wp \\ \widehat{\mathbb{C}} & \xrightarrow{f} & \widehat{\mathbb{C}} \end{array}$$

Note that $f^{-1}([1, \infty]) = [1, \infty] \cup [-1, 0]$ and $\gamma := f^{-1}([-1, 0])$ is a Jordan curve separating the two intervals $[1, \infty]$ and $[-1, 0]$. Set $\mathcal{A} := \widehat{\mathbb{C}} \setminus ([1, \infty] \cup [-1, 0])$. Then $\mathcal{A}^1 := f^{-1}(\mathcal{A}) = \mathcal{A} \setminus \gamma$. So $f : \mathcal{A}^1 \rightarrow \mathcal{A}$ is an exact annular system and then by Theorem 4.5, there are uncountably many wandering components of $\bigcap_{n \geq 0} f^{-n}(\mathcal{A})$ which are Jordan curves.

Let $L \subset \mathbb{C}$ be a line passing through the origin with an irrational slope. Let K be a closed segment on L such that $K \cap \chi(K) = \emptyset$. Denote by π the projection from \mathbb{C} to \mathbb{C}/Λ . Then $\pi : L \rightarrow \pi(L)$ is injective. So $\pi(K)$ is a full wandering continuum

of χ . Moreover, $\wp \circ \pi : L \rightarrow \wp \circ \pi(L)$ is also injective. Therefore $\wp \circ \pi(K) \subset \widehat{\mathbb{C}}$ is a full wandering continuum of f .

We end this section by the following conjecture:

Conjecture: If a postcritically finite rational map has a full wandering continuum it must be a flexible Lattès example.

A Rees-Shishikura's semi-conjugacy

Two postcritically finite branched coverings F and f of $\widehat{\mathbb{C}}$ are said **Thurston-equivalent** if there exists a pair of orientation-preserving homeomorphisms (ϕ_0, ϕ_1) of $\widehat{\mathbb{C}}$ such that ϕ_1 is homotopic to $\phi_0 \text{ rel } \mathcal{P}_F$ and $f \circ \phi_1 = \phi_0 \circ F$.

Theorem A.1. *Let $F : \widehat{\mathbb{C}} \rightarrow \widehat{\mathbb{C}}$ be a postcritically finite branched covering which is Thurston-equivalent to a rational map f through a pair of homeomorphisms (ψ_0, ψ_1) of $\widehat{\mathbb{C}}$. Suppose that F is holomorphic in a neighborhood of the critical cycles of F . Then there exist a neighborhood U of the critical cycles of F and a sequence of homeomorphisms $\{\phi_n\}$ ($n \geq 0$) of $\widehat{\mathbb{C}}$ which are homotopic to $\psi_0 \text{ rel } \mathcal{P}_F$, such that $\phi_n|_U$ is holomorphic, $\phi_n|_U = \phi_0|_U$ and $f \circ \phi_{n+1} = \phi_n \circ F$. The sequence $\{\phi_n\}$ converges uniformly to a continuous map $h : \widehat{\mathbb{C}} \rightarrow \widehat{\mathbb{C}}$. Moreover, the following statements hold:*

- (1) $h \circ F = f \circ h$.
- (2) h is surjective.
- (3) $h^{-1}(w)$ is a single point for $w \in \mathcal{F}_f$ and $h^{-1}(w)$ is either a single point or a full continuum for $w \in \mathcal{J}_f$.
- (4) For points $x, y \in \widehat{\mathbb{C}}$ with $f(x) = y$, $h^{-1}(x)$ is a component of $F^{-1}(h^{-1}(y))$. Moreover, $\deg F|_{h^{-1}(x)} = \deg_x f$.
- (5) $h^{-1}(E)$ is a continuum if $E \subset \widehat{\mathbb{C}}$ is a continuum.
- (6) $h(F^{-1}(E)) = f^{-1}(h(E))$ for any $E \subset \widehat{\mathbb{C}}$.
- (7) $F^{-1}(\widehat{E}) = \widehat{F^{-1}(E)}$ for any $E \subset \widehat{\mathbb{C}}$, where $\widehat{E} = h^{-1}(h(E))$.

This theorem (except (5)-(7)) was proved by Rees and Shishikura for matings of polynomials ([23, 22]). However their proof is still valid in the above more general setting. Here we will only provide a proof for (5)-(7).

Proof. (5). Suppose that $E \subset \widehat{\mathbb{C}}$ is a connected closed subset. The closeness of $h^{-1}(E)$ is easy to see. Now suppose that $h^{-1}(E)$ is not connected, i.e., there are open sets U_1, U_2 in $\widehat{\mathbb{C}}$ such that $h^{-1}(E) \subset U_1 \cup U_2$, $U_1 \cap U_2 = \emptyset$ and both U_1 and U_2 intersect with $h^{-1}(E)$. Then $K := h(\widehat{\mathbb{C}} \setminus (U_1 \cup U_2))$ is a compact set disjoint from E . Since E is connected, there is a connected neighborhood V of E such that $\overline{V} \cap K = \emptyset$. Since $\{\phi_n\}$ converges uniformly to h , there exists an integer $n > 0$ such that

$$d(h, \phi_n) = \sup_{z \in \widehat{\mathbb{C}}} d(h(z), \phi_n(z)) < \min\{d(E, \partial V), d(\overline{V}, K)\},$$

where $d(\cdot, \cdot)$ denotes the spherical distance. Then it follows that $\phi_n(\widehat{\mathbb{C}} \setminus (U_1 \cup U_2)) \cap \overline{V} = \emptyset$, hence $\phi_n^{-1}(V) \subset U_1 \cup U_2$. On the other hand, since $V \supset E$, both U_1 and U_2 intersect with $\phi_n^{-1}(V)$. This contradicts the fact that $\phi_n^{-1}(V)$ is connected.

(6). From $f \circ h(F^{-1}(E)) = h \circ F(F^{-1}(E)) = h(E)$, we have $f(F^{-1}(E)) \subset f^{-1}(h(E))$. Conversely, for any point $w \in f^{-1}(h(E))$, $f(w) \in h(E)$. So there is a

point $z_0 \in E$ such that $f(w) = h(z_0)$. By (5), the map

$$F : h^{-1}(w) \rightarrow h^{-1}(f(w))$$

is surjective. Noticing that $z_0 \in h^{-1}(f(w))$, there is a point $z_1 \in h^{-1}(w)$ such that $F(z_1) = z_0$. So $w = h(z_1) \in h(F^{-1}(z_0)) \subset h(F^{-1}(E))$. Therefore, $f^{-1}(h(E)) \subset h(F^{-1}(E))$.

(7). $F^{-1}(\widehat{E}) = F^{-1}(h^{-1}(h(E))) = h^{-1}(f^{-1}(h(E)))$. From (6), we obtain

$$F^{-1}(\widehat{E}) = h^{-1}(h(F^{-1}(E))) = \widehat{F^{-1}(E)}.$$

□

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